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ALGEBRAIC GEOMETRY III/IV

Problems, set 1.

Exercise 1. Let $C \subset \mathbb{P}^2$ be a curve.

(a) Show that C has finitely many singularities if and only if C has no multiple components.

(b) Show that if D is an irreducible curve in $\mathbb{P}^2$, then $D \subset \mathrm{Sing}(C)$ if and only if D is a multiple component of C.

Exercise 2. Let $P_1, \dots, P_k \in \mathbb{P}^2$ and $S_d(P_1, \dots, P_k)$ denote the $\mathbb{C}$-vectorspace of all homogeneous polynomials $F \in \mathbb{C}[X, Y, Z]$ of degree d vanishing in $P_1, \dots, P_k$.

(a) Let P_1, P_2, P_3, P_4 be four distinct points in $\mathbb{P}^2$, no three of them on a line. Show that

$$\dim S_2(P_1, P_2, P_3, P_4) = 2.$$

(b) Let P_1, P_2, P_3, P_4 be, again, four distinct points. This time, P_1, P_2, P_3 should lie on a common line L and $P_4 \notin L$. Show that we have, again, in this case

$$\dim S_2(P_1, P_2, P_3, P_4) = 2.$$

Exercise 3.

(a) Let C be a non-singular cubic in $\mathbb{P}^2$ and assume for a point $O \in C$ that its tangent meets C with multiplicity three. Show that in this case for every $P \in C$ the three points $O, P, -P$ lie on a projective line (where O is to be considered as the neutral element of the group law).

(b) Let $F(X, Y, Z) = Y^2Z - X^3 + 43XZ^2 - 166Z^3$. Then $C_F \subset \mathbb{P}^2$ is a non-singular cubic. (You don't need to show this!) Show that $O = [0, 1, 0] \in C_F$ has the property introduced in (a). Let $P = [3, 8, 1] \in C_F$. Calculate $-P$.

Exercise 4. Let $C \subset \mathbb{P}^2$ be the cubic defined by $F(X,Y,Z) = 4YZ^2 - X^2Z + Y^3 = 0$. Show that this cubic is non-singular and that for the group law with respect to $O = [1,0,0]$, the point $P = [4,2,1]$ has order four.

Exercise 5. Let $F(X,Y,Z) = Y^2Z + XYZ - X^3 - XZ^2$. Then $C_F \subset \mathbb{P}^2$ is a non-singular cubic. (You don't need to show this!) Let $P = [1,1,1], Q = [1,-1,i] \in C_F$. The addition in C_F is defined with respect to the neutral element $O = [0,1,0] \in C_F$. Calculate $P + Q$.