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ALGEBRAIC GEOMETRY III/IV

Problems, set 4.

Exercise 16.

(a) Let $C = C_F$ be a curve of degree ≥ 3 with $\mathrm{Hess}_F \not\equiv 0$. Let $L \subset \mathbb{P}^2$ be a line which is contained in C with multiplicity $n \geq 1$. Show that L is also contained in Hess_C with multiplicity $\geq 3n-2$.

(b) Let $n \geq 3$ and $F(X,Y,Z) = X^n - Y^{n-1}Z$. Show that $[0,1,0]$ is a flex of $C_F \subset \mathbb{P}^2$ of order $n-2$.

Exercise 17. Locate all flexes of the curve C defined by $F(X,Y,Z) = Y^2Z - X^3 - X^2Z$. Show that all flexes lie on a common projective line. Find all the intersection points of C and Hess_C and calculate their intersection multiplicities $I_P(C, \mathrm{Hess}_C)$. Which of them are singular?

Exercise 18. Let $C \subset \mathbb{P}^2$ be a non-singular cubic and O be a flex of C. Let O be the neutral element of the group law. Prove the following facts:

(a) The points $P, -P, O$ always lie on a line. Another way of putting this is to say that $P + Q = O$ if and only if P, Q and O lie on a common line.

(b) Let P, Q be two points on C. Then the third intersection point of the line through P, Q is $-(P+Q)$.

(c) $P + Q + R = O$ if and only if P, Q and R lie on a common line.

Exercise 19. Let $C \subset \mathbb{P}^2$ be a non-singular cubic and O be a flex of C.

(a) Using Exercise 18, show that a point $P \neq O$ is a flex if and only if P has order 3 with respect to the group law on C (with O as the neutral element).

(b) Let $P, Q \in C$ be two different flexes. Using (b), show that the line through P and Q meets C in a third flex.

Exercise 20. Show that $P = [0, 0, 1]$ is the unique singular point of the curve C determined by $F(X, Y, Z) = Z(X^2 + Y^2) - Y(3X^2 - Y^2)$. Verify that the Hessian curve Hess_C is defined by $G(X, Y, Z) = Z(X^2 + Y^2) + 3Y(3X^2 - Y^2)$. Deduce that F has exactly three collinear flexes.

Exercise 21. Show that the cubic C in $\mathbb{P}^2$ defined by $Y^2Z + XYZ = X^3 + XZ^2$ is non-singular and that $O = [0, 1, 0]$ is a flex. Show that with O being the neutral element of the group law, the point $P = [1, 1, 1]$ has order 4.