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ALGEBRAIC GEOMETRY III/IV

Problems, set 5.

Please hand in your solutions to the exercises **17, 23** on **Thursday, 16 March**, in the lecture.

Exercise 22. Explain why the union of two distinct projective lines $L_1 \cup L_2 \subset \mathbb{P}^2$ cannot be a Riemann surface.

Exercise 23. In this exercise we derive the *Weierstraß normal form* of a cubic. Let $C \subset \mathbb{P}^2$ be a non-singular cubic defined by the polynomial $F \in \mathbb{C}[X, Y, Z]$.

(a) Reason that C has a flex P of order one. By change of coordinates we may arrange that $P = [0, 1, 0]$, and that the tangent of C at P is given by $Z = 0$. Conclude that F has the form

$$F(X,Y,Z) = aX^3 + bX^2Z + cXYZ + dXZ^2 + eY^2Z + fYZ^2 + gZ^3,$$

with $a \neq 0$, $e \neq 0$.

(b) Show that the substitution of Y by $Y - \frac{c}{2e}X - \frac{f}{2e}Z$ implies vanishing of the coefficients of XYZ and YZ^2, and that no new non-zero terms are generated. So, another change of variables yields

$$F(X,Y,Z) = a'X^3 + b'X^2Z + d'XZ^2 + e'Y^2Z + gZ^3,$$

still with $a', e' \neq 0$.

(c) Show that substitution of X by $X - \frac{b'}{3a'}Z$ yields the equation

$$F(X,Y,Z) = a''X^3 + d''XZ^2 + e''Y^2Z + g''Z^3,$$

still with $a'', e'' \neq 0$.

(d) Argue, why we can, after another coordinate change, obtain the final equation
$$Y^2Z = 4X^3 - \alpha XZ^2 - \beta Z^3,$$
for the cubic C.

Additional remarks to Exercise 23: It can be shown that the cubic $Y^2Z = 4X^3 - \alpha XZ^2 - \beta Z^3$ is non-singular iff $\alpha^3 - 27\beta^2 \neq 0$. The function $j = \frac{\alpha^3}{\alpha^3 - 27\beta^2}$ turns out to be a projective invariant of the Weierstraß normal form. Two normal forms $Y^2Z = 4X^3 - \alpha XZ^2 - \beta Z^3$ are projectively equivalent iff the corresponding values of j coincide. In particular, there are uncountably many projectively non-equivalent non-singular cubics. The final classification of all cubics (non-singular and singular) looks as follows:

(a) Every non-singular cubic is projectively equivalent to a curve of the type $Y^2Z = 4X^3 - \alpha XZ^2 - \beta Z^3$.

(b) Every irreducible singular cubic is projectively equivalent to the curve $X^3 + Y^3 - XYZ = 0$ (cubic with a nodal singularity) or to the curve $X^3 - Y^2Z = 0$ (cubic with a cuspidal singularity).

(c) Every reducible cubic C is either a conic plus a chord, a conic plus a tangent line, or C consists of three lines L_1, L_2, L which meet in three different points (triangle), in one common point (triple point), or two or three of the lines L_j coincide.