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ALGEBRAIC GEOMETRY III/IV
Solutions to the exercises

Exercise 1.

(a) Let $C = C_F$ with $F = G^2H$. Then we have $F_X = G(2G_XH + GH_X)$, so both F and F_X vanish on C_G. Similarly, F_Y and F_Z vanish on C_G and $\mathrm{Sing}(F) \supset C_G$. It remains to show that if F has no multiple components then $\mathrm{Sing}(F)$ is finite.

Let $C = C_{F_1} \cup \dots C_{F_n}$ and the C_{F_j} are the pairwise different irreducible components of C. Then we know

$$\mathrm{Sing}(C_F) = \bigcup_j \mathrm{Sing}(C_{F_j}) \cup \left(\bigcup_{i<j} C_{F_i} \cap C_{F_j} \right).$$

Since F_j is irreducible, $\mathrm{Sing}(C_{F_j})$ is finite. If $C_{F_i} \cap C_{F_j}$ would be infinite, then both curves would have a common component (by weak Bezout). This cannot be since C_{F_i} and C_{F_j} are both irreducible and different.

(b) If D is a multiple component of C then it is clear by similar arguments as in (a) that $D \subset \mathrm{Sing}(C)$. Assume that D is irreducible and $D \subset \mathrm{Sing}(C)$. Let $C = C_F$ with multiplicities and $F = F_1 \dots F_n$ be the irreducible factors of F with multiplicities. Note that

$$\mathrm{Sing}(C_F) = \bigcup_j \mathrm{Sing}(C_{F_j}) \cup \left(\bigcup_{i<j} C_{F_i} \cap C_{F_j} \right).$$

Since $D \subset \mathrm{Sing}(C_F)$ and since $\mathrm{Sing}(C_{F_j})$ is finite, there must be a $i < j$ such that $D \cap C_{F_i} \cap C_{F_j}$ is infinite. Since D, C_{F_i} and C_{F_j} are irreducible, we conclude with weak Bezout that $D = C_{F_i} = C_{F_j}$. So D is a multiple component of C.

Exercise 2. We first obtain

$$\dim S_2(P_1, \dots, P_4) \geq \binom{4}{2} - 4 = 6 - 4 \geq 2.$$

So we only have to prove in both cases that $\dim S_2(P_1,\ldots,P_4) \geq 3$ leads to a contradiction.

(a) Choose a line L through P_4 which doesn't contain P_1, P_2, P_3. Choose two more points $P_5, P_6 \in L$. $\dim S_2(P_1,\ldots,P_4) \geq 3$ would imply that there exists a conic W through $P_1,\ldots,P_6$. Since P_4, P_5, P_6 lie on a line, this conic W has to be singular, i.e., $W = L \cup L'$ (L' not necessarily different from L.) Since $P_1, P_2, P_3 \notin L$, we must have $P_1, P_2, P_3 \in L'$, in contradiction to the assumption of (a).

(b) Let $H \in \mathbb{C}[X,Y,Z]$ be a non-trivial polynomial defining the line L through P_1, P_2, P_3. Then any non-trivial $F \in S_2(P_1,P_2,P_3,P_4)$ defines a reducible conic $C_F = L \cap L'$. Therefore, F is divisible by H and we have an injective linear map

$$S_2(P_1,P_2,P_3,P_4) \to S_1(P_4), \quad F \mapsto F/H.$$

So the assumption $\dim S_2(P_1,\ldots,P_4) \geq 3$ would lead to

$$\dim S_1(P_4) \geq 3.$$

But the space of all linear polynomials $G(X,Y,Z) = aX+bY = CZ$ vanishing in a point P_4 is clearly of dimension 2. So we arrive at a contradiction.

Another solution for (a) (there is a similar alternative solution for (b)): P_1, P_2, P_3, P_4 are in general position. By a change of coordinates, we can assume that $P_1 = [1,0,0], P_2 = [0,1,0], P_3 = [0,0,1]$ and $P_4 = [1,1,1]$. Assume that

$$F(X,Y,Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \in S_2(P_1,\ldots,P_4).$$

Then $F(P_1) = F(P_2) = F(P_3) = 0$ implies that $a = b = c = 0$. So we have for $F(X,Y,Z) \in S_2(P_1,P_2,P_3,P_4)$:

$$F(X,Y,Z) = dXY + eXZ + fYZ.$$

The last condition $F(P_4) = 0$ implies that $d+e+f = 0$, so $S_2(P_1,P_2,P_3,P_4)$ can be identified with the 2-dimensional complex vector space

$$\{ \begin{pmatrix} d \\ e \\ f \end{pmatrix} \in \mathbb{C}^3 \mid d+e+f = 0 \}.$$

Exercise 3.

(a) If the tangent L of C at O meets C at O with multiplicity three, then we have $L \cap C = \{O, O, O\}$. So the third point of intersection of this tangent is, again, O. So, by the law to obtain the inverse $-P$, we have to connect P and O by a projective line and $-P$ is the third point of this line. This shows that $O, P, -P$ lie always on a projective line in this case. (The condition of the exercise is equivalent to the fact that O is a flex).

(b) We have

$$\begin{aligned} F_X(X,Y,Z) &= -3X^2 + 43Z^2, \\ F_Y(X,Y,Z) &= 2YZ, \\ F_Z(X,Y,Z) &= Y^2 + 86XZ - 498Z^2. \end{aligned}$$

Then the tangent of C_F at $O = [0,1,0]$ is given by $Z = 0$. Plugging this into F yields

$$F(X,Y,0) = -X^3,$$

so the tangent meets C at $O = [0,1,0]$ with multiplicity three. The projective line L through $P = [3,8,1]$ and $O = [0,1,0]$ is given by $X = 3Z$. We calculate the third intersection point of $L \cap C$ by plugging this into F:

$$F(3Z,Y,Z) = Y^2Z - 27Z^3 + 129Z^3 - 166Z^3 = Y^2Z + 64Z^3 = Z(Y-8Z)(Y+8Z).$$

So the third point of intersection is given by $X = 3Z$ and $Y + 8Z = 0$, i.e., $-P = [3,-8,1]$.

Exercise 4. We have $F_X = -2XZ$, $F_Y = 4Z^2 + 3Y^2$ and $F_Z = 8YZ + X^2$. $F_X = 0$ implies $X = 0$ or $Z = 0$. $Z = 0$ and $F_Y = F_Z = 0$ leads immediately to $Y = 0$ and $X = 0$. So $X = 0$. But $F_Z = 0$ implies that $YZ = 0$, thus $Y = 0$. But $X = 0$ and $Y = 0$ and $F_Y = 0$ implies also $Z = 0$. So there is no singular point.

Let $P = [4,2,1]$. We want to calculate $2P$. The tangent L_1 at C in P is given by the equation

$$X - 2Y = 0.$$

To calculate the third point of the intersection $L_1 \cap C$ we plug $X = 2Y$ into F:

$$F(2Y,Y,Z) = Y(2Z-Y)^2 = 0.$$

The point given by $X = 2Y$, $Y = 2Z$, namely P, has multiplicity two (clear, since L_1 is the tangent) and the point $X = 2Y$, $Y = 0$, namely $[0,0,1]$, has multiplicity one and is the third point. The line L_2 through O and $[0,0,1]$ is given by $Y=0$. To calculate the third intersection point of L_2 with C we plug in $Y=0$ in F:

$$F(X,0,Z) = -X^2Z = 0.$$

The point given by $Y=0$, $X=0$, namely $[0,0,1]$ has multiplicity two and the point given by $Y=0$, $Z=0$, namely $[1,0,0]$ has multiplicity one. So the third intersection point is $2P = [0,0,1] \neq O$.

Next, we want to calculate $4P$: The tangent line L_3 at C in $2P$ is given by $Y=0$. To calculate the third intersection point of L_3 with C we plug in $Y=0$ in F and obtain again

$$F(X,0,Z) = -X^2Z = 0.$$

So the third intersection point is $O=[1,0,0]$ in this time. We need to choose the line through O and O, namely the tangent of C at O and find the third intersection point to find $4P$: The tangent of C at O is $Z=0$. Now,

$$F(X,Y,0) = Y^3 = 0,$$

so the point $Z=0$, $Y=0$, namely $O=[1,0,0]$, has multiplicity three. So the third intersection point in this case is $O=[1,0,0]=4P$.

Exercise 5. The projective line L through $P=[1,1,1]$ and $Q=[1,-1,i]$ is given by

$$0 = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & i \\ X & Y & Z \end{pmatrix} = (1+i)X + (1-i)Y - 2Z.$$

To calculate the third intersection point of $L \cap C$ we plug

$$Z = \frac{1+i}{2}X + \frac{1-i}{2}Y$$

into F and obtain

$$\begin{aligned}
&F(X,Y,\frac{1+i}{2}X+\frac{1-i}{2}Y)\\
&\quad=(Y^2+XY)\left(\frac{1+i}{2}X+\frac{1-i}{2}Y\right)-X^3-X\left(\frac{1+i}{2}X+\frac{1-i}{2}Y\right)^2\\
&\qquad\quad=XY^2+\frac{1-i}{2}Y^3+\frac{1+i}{2}X^2Y-X^3-\left(\frac{i+1}{2}\right)^2X^3\\
&\qquad\qquad\quad-\left(\frac{1-i}{2}\right)^2XY^2-\frac{(1+i)(1-i)}{2}X^2Y\\
&\qquad\qquad\qquad=\frac{-2-i}{2}X^3+\frac{-1+i}{2}X^2Y+\frac{2+i}{2}XY^2+\frac{1-i}{2}Y^3.
\end{aligned}$$

Since P and Q are intersection points, $X-Y$ and $X+Y$ must be factors of this polynomial, and we therefore make the Ansatz

$$\begin{aligned}
\frac{-2-i}{2}X^3+\frac{-1+i}{2}X^2Y+\frac{2+i}{2}XY^2+\frac{1-i}{2}Y^3&=\frac{-2-i}{2}(X-Y)(X+Y)(X+aY)\\
&=\frac{-2-i}{2}(X^2-Y^2)(X+aY).
\end{aligned}$$

This leads to

$$\frac{2+i}{2}a=\frac{1-i}{2},$$

i.e.,

$$a=\frac{1-i}{2+i}=\frac{1-3i}{5}.$$

We easily check that

$$\frac{-2-i}{2}X^3+\frac{-1+i}{2}X^2Y+\frac{2+i}{2}XY^2+\frac{1-i}{2}Y^3=\frac{-2-i}{2}(X^2-Y^2)(X+\frac{1-3i}{5}Y).$$

So the third intersection point R of $L\cap C$ is given by

$$Z=\frac{1+i}{2}X+\frac{1-i}{2}Y,\quad X=\frac{-1+3i}{5}Y,$$

i.e.,

$$R=\left[\frac{-1+3i}{5},1,\frac{1-3i}{10}\right].$$

The projective line through R and $O = [0, 1, 0]$ is obviously given by

$$X = -2Z,$$

so plugging this into F we obtain

$$F(-2Z, Y, Z) = Y^2 Z - 2YZ^2 + 10Z^3 = Z(Y^2 - 2YZ + 10Z^2) = 0.$$

$Z = 0$ corresponds to the intersection point $[0, 1, 0]$ and the other two points are solutions of $Y^2 - 2YZ + 10Z^2$. Setting $Y = 1$ we obtain the affine equation

$$1 - 2Z + 10Z^2 = 0$$

with solutions

$$Z = \frac{1 \pm 3i}{10},$$

so the other two intersection points are

$$R = \left[\frac{-1+3i}{5}, 1, \frac{1-3i}{10}\right] \quad \text{and} \quad \left[\frac{-1-3i}{5}, 1, \frac{1+3i}{10}\right].$$

So we end up with

$$P + Q = \left[\frac{-1-3i}{5}, 1, \frac{1+3i}{10}\right].$$

Exercise 6.

(a) Writing $F(X,Y,Z) = X^2 + Y^2 - 2YZ - 3Z^2$ and $G(X,Y,Z) = X^2 - 2YZ - 2Z^2$ yields

$$R(X,Y) = \mathrm{Res}_Z(F(X,Y,Z), G(X,Y,Z)) = \det \begin{pmatrix} X^2+Y^2 & -2Y & -3 & 0 \\ 0 & X^2+Y^2 & -2Y & -3 \\ X^2 & -2Y & -2 & 0 \\ 0 & X^2 & -2Y & -2 \end{pmatrix}$$

$$= \det \begin{pmatrix} X^2+Y^2 & -2Y & -3 & 0 \\ 0 & -\frac{1}{2}X^2+Y^2 & Y & 0 \\ X^2 & -2Y & -2 & 0 \\ 0 & X^2 & -2Y & -2 \end{pmatrix}$$

$$= -2\det \begin{pmatrix} X^2+Y^2 & -2Y & -3 \\ 0 & -\frac{1}{2}X^2+Y^2 & Y \\ X^2 & -2Y & -2 \end{pmatrix} = X^4 - 4X^2Y^2 = X^2(X-2Y)(X+2Y).$$

(b) We have $F(0,0,1) = -3$ and $G(0,0,1) = -2$. Assume that $[X,Y,0]$ is an intersection point. Then $F(X,Y,0) = X^2+Y^2 = 0$ and $G(X,Y,0) = X^2$ yield $X = Y = 0$, a contradiction. So the intersection points are of the form $[X,Y,1]$. So such an intersection point satisfies

$$0 = F(X,Y,1) = X^2 + Y^2 - 2Y - 3 = G(X,Y,1) = X^2 - 2Y - 2.$$

This enables us to eliminate X^2 and to obtain

$$0 = (X^2 + Y^2 - 2Y - 3) - (X^2 - 2Y - 2) = Y^2 - 1.$$

So we arrive at two possibilities: $Y = -1$ and $Y = 1$. The case $Y = 1$ leads to $X = \pm 2$, and $[2,1,1]$ and $[-2,1,1]$ are actually intersection points. The case $Y = -1$ leads to $X = 0$, and $[0,-1,1]$ is also an intersection point. So we obtained

$$C \cap D = \{P_1 = [2,1,1], P_2 = [-2,1,1], P_3 = [0,-1,1]\}.$$

The line through P_1, P_2 is given by $Y = Z$ and $[0,0,1]$ doesn't lie on this line. The line through P_1, P_3 is given by $X - Y - Z = 0$ and $[0,0,1]$ doesn't lie on this line. Finally, the line through P_2, P_3 is given by $X + Y + Z = 0$ and $[0,0,1]$ doesn't lie on this line.

(c) By the "theory behind the scenery" we know that each of the three different intersection points lies on a corresponding line contained in $R(X,Y) =$

$X^2(X-2Y)(X+2Y)$, and the multiplicity of this line is the intersection multliplicity of the corresponding point. So we conclude that

$$I_R(C,D)=\begin{cases}2 & \text{if } R=P_3,\\ 1 & \text{if } R=P_1 \text{ or } R=P_2,\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 7. Plugging $Z=2Y$ into F we obtain

$$F(X,Y,2Y)=Y^2(4X-Y)=0.$$

The solution $Y=0$ leads to the intersection point $[1,0,0]$ and the solution $Y=4X$ to the intersection point $[1,4,8]$. The intersection multiplicity of $[1,0,0]$ in $L\cap C_F$ is two and the intersection multiplicity of $[1,4,8]$ in $L\cap C_F$ is one.

Using "modern techniques", we have

$$\begin{aligned}I_R(L,C_F)&=I_R(Z-2Y,XZ^2-Y^3-XZ(Z-2Y))=I_R(Z-2Y,2XYZ-Y^3)\\&=I_R(Z-2Y,Y)+I_R(Z-2Y,2XZ-Y^2)\\&=I_R(Z,Y)+I_R(Z-2Y,2XZ-Y^2-2X(Z-2Y))\\&=I_R(Z,Y)+I_R(Z-2Y,4XY-Y^2)\\&=I_R(Z,Y)+I_R(Z-2Y,Y)+I_R(Z-2Y,4X-Y)=2I_R(Z,Y)+I_R(Z-2Y,4X-Y).\end{aligned}$$

So the intersection points are $R=[1,0,0]$ with multiplicity two and $R=[1,4,8]$ with multiplicity one.

Exercise 8. We have

$$\begin{aligned}I_R(G,F-ZG)&=I_R(G,2X^2Z+2Y^2Z-X^3)\\&=I_R(G,2X^2Z+2Y^2Z-X^3-XG)=I_R(G,2Y^2Z+XY^2)\\&=2I_R(G,Y)+I_R(G,2Z+X)=2I_R(2XZ-X^2,Y)+I_R(2Z+X,G)\\&=2I_R(X,Y)+2I_R(2Z-X,Y)+I_R(2Z+X,G-X(2Z+X))\\&=2I_R(X,Y)+2I_R(2Z-X,Y)+I_R(2Z+X,Y^2+2X^2)\\&=2I_R(X,Y)+2I_R(2Z-X,Y)+I_R(2Z+X,(Y+i\sqrt{2}X)(Y-i\sqrt{2}X))\\&=2I_R(X,Y)+2I_R(2Z-X,Y)+I_R(2Z+X,Y+i\sqrt{2}X)+I_R(2Z+X,Y-i\sqrt{2}X).\end{aligned}$$

So we have the intersection points $[0,0,1]$ and $[2,0,1]$ with multiplicity two and $[2, \pm 2i\sqrt{2}, -1]$ with multiplicity one.

Exercise 9. We have $G(X,Y,Z) - Z^2F(X,Y,Z) = -Y^6 + Y^4Z^2 = Y^4(Z^2 - Y^2)$. So we obtain

$$\begin{aligned} I_R(F,G) = I_R(F, G - Z^2F) &= 4I_R(Z^4 + X^2Z^2, Y) + I_R(F, Z-Y) + I_R(F, Z+Y) \\ &= 4I_R(Z^2 + X^2, Y) + 8I_R(Z,Y) + I_R(F, Z-Y) + I_R(F, Z+Y). \end{aligned}$$

Now, $F(X,Y,Z) = (Z-Y)(Z+Y)(Z^2+Y^2) + X^2Z^2$, so

$$\begin{aligned} I_R(F,G) &= 4I_R(Z^2+X^2, Y) + 8I_R(Z,Y) + I_R(X^2Z^2, Z-Y) + I_R(X^2Z^2, Z+Y) \\ &= 4I_R(Z-iX, Y) + 4I_R(Z+iX, Y) + 12I_R(Z,Y) + 2I_R(X, Z-Y) + 2I_R(X, Z+Y). \end{aligned}$$

So $[1,0,i]$ and $[1,0,-i]$ both have multiplicity 4, $[1,0,0]$ has multiplicity 12 and both $[0,1,1]$ and $[0,1,-1]$ have multiplicity 2.

Exercise 10. We have

$$I_R(F,G) = I_R(F, G-F) = I_R(F, Z^2 - 4XZ) = I_R(X^2+Y^2, Z) + I_R(F, Z-4X).$$

To determine $I_R(F, Z-4X)$ we plug $Z - 4X$ into F:

$$0 = F(X,Y,4X) = Y^2 - 15X^2 = (Y - \sqrt{15}X)(Y + \sqrt{15}X).$$

So we obtain the two intersection points $[1, \sqrt{15}, 4]$ and $[1, -\sqrt{15}, 4]$, both with multiplicity one. $I_R(X^2 + Y^2, Z)$ yields the two intersection points $[1, i, 0]$ and $[1, -i, 0]$, both again with multiplicity one.

Exercise 11. We have

$$\begin{aligned} F_X &= 3X^2 - YZ, \\ F_Y &= 3Y^2 - XZ, \\ F_z &= -XY. \end{aligned}$$

Therefore, we obtain

$$G(X,Y,Z) = \det \begin{pmatrix} 6X & -Z & -Y \\ -Z & 6Y & -X \\ -Y & -X & 0 \end{pmatrix} = -2(3X^2 + 3Y^2 + XYZ).$$

We conclude that

$$
\begin{aligned}
&I_P(F,G) = I_P(X^3+Y^3-XYZ, 3X^3+3Y^3+XYZ)\\
&= I_P(X^3+Y^3, 3X^3+3Y^3+XYZ) = I_P(X^3+Y^3, 3X^3+3Y^3+XYZ)\\
&= I_P(X^3+Y^3, XYZ) = I_P(X+Y,X) + I_P(X+Y,Y) + I_P(X+Y,Z)\\
&\qquad + I_P(X+\zeta_3 Y, X) + I_P(X+\zeta_3 Y, Y) + I_P(X+\zeta_3 Y, Z)\\
&\qquad + I_P(X+\zeta_3^2 Y, X) + I_P(X+\zeta_3^2 Y, Y) + I_P(X+\zeta_3^2 Y, Z)\\
&\quad = 6I_P(Y,X) + I_P(X+Y,Z) + I_P(X+\zeta_3 Y, Z) + I_P(X+\zeta_3^2 Y, Z).
\end{aligned}
$$

The intersection points are therefore $P_0 = [0,0,1]$ with multiplicity six and all three points $[-1,1,0], [-\zeta_3,1,0], [-\zeta_3^2,1,0]$ with multiplicity one. $[0,0,1]$ is a singularity and the other three points are non-singular since we have for them $F_Y(X,1,0) = 3$.

Exercise 12. We have

$$\text{non-singular} \quad \Rightarrow \quad \text{irreducible} \quad \Rightarrow \quad \text{no multiple compontents}$$

but not the reverse directions.

An example of an irreducible singular curve is, e.g., C_F with

$$F(X,Y,Z) = Y^2Z - X^3.$$

Because of $F_X = -3X^2$, $F_Y = 2YZ$, $F_Z = Y^2$, the only singularity of C_F is $[0,0,1]$. If C_F were reducible, we would have $F = GH$, with a non-vanishing quadratic term G and a linear term H. Since two curves C_G and C_H always meet, and they must meet in a singularity of C_F, we conclude that $[0,0,1] \in C_H$, so $H(X,Y) = \alpha X + \beta Y$. In the Ansatz

$$(\alpha X + \beta Y)G(X,Y,Z) = -X^3 + Y^2 Z$$

we conclude that the coefficient of X^2 in G must be $-\frac{1}{\alpha}$ and the coefficient of YZ in G must be $\frac{1}{\beta}$. We see that $\alpha, \beta \neq 0$. Moreover, the coefficient of XZ in G must be zero, for otherwise $(\alpha X + \beta Y)G(X,Y,Z)$ would contain a non-vanishing X^2Z term. But

$$
\begin{aligned}
&(\alpha X + \beta Y)(-\frac{1}{\alpha}\cdot X^2 + \frac{1}{\beta}\cdot YZ + 0\cdot XZ + \text{other terms})\\
&\qquad\qquad\qquad = -X^3 + \frac{\alpha}{\beta}XYZ + Y^2Z + \text{other terms},
\end{aligned}
$$

and we can't get rid of $\frac{\alpha}{\beta}XYZ \neq 0$. So C_F is irreducible.

An example of a reducible curve without multiple components is C_F with $F(X,Y,Z) = XY$, where the different components are $X = 0$ and $Y = 0$.

Exercise 13. We have

$$F(X,Y,Z) = Z^4 + X^2Z^2 - Y^4, \quad G(X,Y,Z) = Z^6 + X^2Z^4 - Y^6.$$

For $P = [1,0,0]$ we have to write F and G as polynomials in X-powers and obtain

$$\begin{aligned} F(X,Y,Z) &= a_2(Y,Z)X^2 + a_4(Y,Z) = Z^2X^2 + (Z^4 - Y^4), \\ G(X,Y,Z) &= b_4(Y,Z)X^2 + b_6(Y,Z) = Z^4X^2 + (Z^6 - Y^6). \end{aligned}$$

This immediately implies $m_P(C_F) = 2$ and $m_P(C_G) = 4$. Since

$$m_P(C_F) \cdot m_P(C_G) = 8 < 12 = I_P(C_F, C_G),$$

there must be coinciding tangents. To find the tangent lines of C_F at P we have to look at $a_2(Y,Z) = Z^2$. So C_F has the tangent line $Z = 0$ with multiplicity 2. To find the tangent lines of C_G at P we have to look at $b_4(Y,Z) = Z^4$. So C_G has the tangent line $Z = 0$ with multiplicity 4.

For $P = [1,0,\pm i]$ we immediately see that $F_X(P) \neq 0$ and that $G_X(P) \neq 0$. So, we conclude that $m_P(C_F) = m_P(C_G) = 1$. Since

$$m_P(C_F) \cdot m_P(C_G) = 1 < 4 = I_P(C_F, C_G),$$

the tangent lines of C_F and C_G at P must coincide. The tangent line of C_F at P is given by

$$0 = F_X(P)X + F_Y(P)Y + F_Z(P)Z = -2X \mp 2iZ,$$

which simplifies to $X \pm iZ = 0$. The tangent line of C_G at P is given by

$$0 = G_X(P)X + G_Y(P)Y + G_Z(P)Z = 2X \pm 2iZ,$$

which simplifies, again, to $X \pm iZ = 0$. So the tangent lines coincide.

For $P = [0,1,\pm 1]$ we immediately see that $F_Y(P) \neq 0$ and that $G_Y(P) \neq 0$. So, we conclude that $m_P(C_F) = m_P(C_G) = 1$. Since

$$m_P(C_F) \cdot m_P(C_G) = 1 < 2 = I_P(C_F, C_G),$$

the tangent lines of C_F and C_G at P must coincide. The tangent line of C_F at P is given by

$$0 = F_X(P)X + F_Y(P)Y + F_Z(P)Z = -4Y \pm 4Z,$$

which simplifies to $Y \mp Z = 0$. The tangent line of C_G at P is given by

$$0 = G_X(P)X + G_Y(P)Y + G_Z(P)Z = -6Y \pm 6Z,$$

which simplifies, again, to $Y \mp Z = 0$. So the tangent lines coincide.

Exercise 14. Since

$$F(X, Y, Z) = X^3 + 3X^2Z + XY^2 - XZ^2 + Y^2Z - 3Z^3,$$

we obtain

$$F(1, \pm 2, -1) = 1 - 3 + 4 - 1 - 4 + 3 = 0.$$

The partial derivatives are given by

$$\begin{aligned} F_X(X, Y, Z) &= 3X^2 + 6XZ + Y^2 - Z^2, \\ F_Y(X, Y, Z) &= 2XY + 2YZ, \\ F_Z(X, Y, Z) &= 3X^2 - 2XZ + Y^2 - 9Z^2. \end{aligned}$$

So we obtain

$$\begin{aligned} F_X(1, \pm 2, -1) &= 3 - 6 + 4 - 1 = 0, \\ F_Y(1, \pm 2, -1) &= \pm 4 \mp 4 = 0, \\ F_Z(1, \pm 2, -1) &= 3 + 2 + 4 - 9 = 0, \end{aligned}$$

which shows that $[1, \pm 2, -1]$ are both singularities of the cubic C_F. An irreducible curve C of degree d has $\leq \frac{(d-1)(d-2)}{2}$ singularities. Therefore, if C_F were irreducible, it would have $\leq \frac{2 \cdot 1}{2} = 1$ singularities. Since C_F has two singularities, is must be reducible. (In fact, we have $F(X, Y, Z) = (X + Z)(X^2 + 2XZ + Y^2 - 3Z^2)$.)

Exercise 15.

(a) An irreducible curve of degree d has at most $\frac{(d-1)(d-2)}{2}$ singularities. So $d = 4$ yields $\frac{3 \cdot 2}{2} = 3$ as an upper bound for the number of singularities. So an irreducible quartic can have at most three singularities. So the answer is NO!

(b) Using the formula

$$\sum_{j=1}^{6} m_{P_j}(C)(m_{P_j} - 1) \leq d(d-1),$$

yields

$$\sum_{j=1}^{6} m_{P_j}(C)(m_{P_j} - 1) = 2 \cdot 2 + 3 \cdot 6 + 12 = 34 \leq d(d-1).$$

Since $d \leq 6$ doesn't satisfy this inequality, we must have $d \geq 7$.

(c) A curve C of degree d without multiple components satisfies

$$\sum_{P \in C} m_P(C)(m_P(C) - 1) \leq d(d-1).$$

So one triple point and three double points lead to

$$\sum_{P \in C} m_P(C)(m_P(C) - 1) = 3 \cdot 2 + 3 \cdot 2 = 12,$$

and $d = 4$ also leads to $d(d-1) = 12$, so our formula doesn't rule out the existence of such a quartic. In fact, if we choose three different projective lines L_1, L_2, L_3 through the same point P and a fourth one L_4 not through P, then $L_1 \cup L_2 \cup L_3 \cup L_4$ is a quartic without multiple components having precisely one triple point and three double points. This shows existence of such a curve.

Exercise 16.

(a) W.l.o.g. we can assume that $L : Z = 0$ and $n \geq 2$ (the case $n = 1$ was discussed in the lecture). Then we have $F(X, Y, Z) = Z^n G(X, Y, Z)$ and

$$\begin{aligned}
&\mathrm{Hess}_F \\
&= Z^{3n-4} \det \begin{pmatrix} ZG_{XX} & ZG_{XY} & Z^2G_{XZ} + nZG_X \\ ZG_{YX} & ZG_{YY} & Z^2G_{YZ} + nZG_Y \\ ZG_{ZX} + nG_X & ZG_{ZY} + nG_Y & Z^2G_{ZZ} + 2nZG_Z + n(n-1)G \end{pmatrix} \\
&= Z^{3n-3} G_{XX} \det \begin{pmatrix} ZG_{YY} & Z^2G_{YZ} + nZG_Y \\ * & * \end{pmatrix} \\
&\quad - Z^{3n-3} G_{YX} \det \begin{pmatrix} ZG_{XY} & Z^2G_{XZ} + nZG_X \\ * & * \end{pmatrix} \\
&\quad + Z^{3n-4}(ZG_{ZX} + nG_X) \det \begin{pmatrix} ZG_{XY} & Z^2G_{XZ} + nZG_X \\ ZG_{YY} & Z^2G_{YZ} + nZG_Y \end{pmatrix} \\
&= Z^{3n-2} G_{XX} \det \begin{pmatrix} G_{YY} & ZG_{YZ} + nG_Y \\ * & * \end{pmatrix} \\
&\quad - Z^{3n-2} G_{YX} \det \begin{pmatrix} G_{XY} & ZG_{XZ} + nZG_X \\ * & * \end{pmatrix} \\
&\quad + Z^{3n-2}(ZG_{ZX} + nG_X) \det \begin{pmatrix} G_{XY} & ZG_{XZ} + nG_X \\ G_{YY} & ZG_{YZ} + nG_Y \end{pmatrix}.
\end{aligned}$$

This proves that L is contained with multiplicity $\geq 3n - 2$ in Hess_C.

(b) The case $n = 3$ was discussed in the lecture. So, we assume that $n \geq 4$. We have $F_X = nX^{n-1}$, $F_Y = -(n-1)Y^{n-2}Z$, $F_Z = -Y^{n-1}$. So $[0, 1, 0] \in C_F$ but $[0, 1, 0] \notin C_{F_Z}$, so $[0, 1, 0]$ is a non-singular point of C_F. The Hessian is given by

$$\mathrm{Hess}_F = \det \begin{pmatrix} n(n-1)X^{n-2} & 0 & 0 \\ 0 & * & -(n-1)Y^{n-2} \\ 0 & -(n-1)Y^{n-2} & 0 \end{pmatrix}.$$

Plugging $[0, 1, 0]$ into Hess_F yields

$$\mathrm{Hess}_F(0, 1, 0) = \det \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & -(n-1) \\ 0 & -(n-1) & 0 \end{pmatrix} = 0.$$

So $[0, 1, 0] \in C_F$ is a flex. We can write F in the form

$$F(X, Y, Z) = ZG(X, Y, Z) + X^{r+2}H(X, Y) = ZY^{n-1} + X^n,$$

and $r = n - 2 \geq 1$ is the order of the flex.

Exercise 17. We have $F_X = -3X^2 - 2XZ, F_Y = 2YZ, F_Z = Y^2 - X^2$ and

$$\begin{aligned}\mathrm{Hess}_F = \det \begin{pmatrix} -6X - 2Z & 0 & -2X \\ 0 & 2Z & 2Y \\ -2X & 2Y & 0 \end{pmatrix} \\ = -8X^2Z + 8Y^2(3X + Z) = 8(3XY^2 - X^2Z + Y^2Z).\end{aligned}$$

$F = \mathrm{Hess}_F = 0$ leads to

$$0 = \frac{1}{8}\mathrm{Hess}_F - F = X(X^2 - 3Y^2),$$

i.e., $X = 0$ or $Y = \frac{\pm i}{\sqrt{3}}X$.

$X = 0$ leads to $F(0, Y, Z) = Y^2Z = 0$, i.e., $Y = 0$ or $Z = 0$. The point $[0, 0, 1]$ is singular, so it cannot be a flex. The point $[0, 1, 0]$ is non-singular since $F_Z(0, 1, 0) = 1$, so it is a flex. The other candidates for flexes are of the form $[X, Y, Z] = [1, \frac{\pm i}{\sqrt{3}}, Z]$. We have

$$F(1, \frac{\pm i}{\sqrt{3}}, Z) = \left(\frac{\pm i}{\sqrt{3}}\right)^2 Z - 1 - Z = -\frac{4}{3}Z - 1,$$

so $F(1, \frac{\pm i}{\sqrt{3}}, Z) = 0$ implies $Z = -\frac{3}{4}$. So the other two points satisfying $F = \mathrm{Hess}_F = 0$ are

$$[X, Y, Z] = [1, \frac{\pm i}{\sqrt{3}}, -\frac{3}{4}].$$

Since F_Y doesn't vanish at these points, they are flexes. The three flexes are lying on the projective line $3X + 4Z = 0$.

Next, we investigate on the intersection multiplicities. We have

$$\begin{aligned}I_P(F, \mathrm{Hess}_F) &= I_P(F, \mathrm{Hess}_F - F) = I_P(Y^2Z - X^3 - X^2Z, X^3 + 3XY^2) \\ &= I_P(Y^2Z, X) + I_P(F, X^2 + 3Y^2) \\ &= 2I_P(Y, X) + I_P(Z, X) + I_P(F, X + i\sqrt{3}Y) + I_P(F, X - i\sqrt{3}Y).\end{aligned}$$

The intersection multiplicities of the intersection points of the curve C and the line $X + i\sqrt{3}Y = 0$ are calculated as follows: $X + i\sqrt{3}Y = 0$ is equivalent to $Y = \frac{i}{\sqrt{3}}X$. Plugging this into F we obtain

$$F(X, \frac{i}{\sqrt{3}}X, Z) = -X^2(X + \frac{4}{3}Z) = 0.$$

So the intersection points are $[0,0,1]$ with multiplicity 2 and $[1, \frac{i}{\sqrt{3}}, -\frac{3}{4}]$ with multiplicity 1. Similarly, we obtain for the intersection points of C and $X - i\sqrt{3}Y = 0$: the point $[0,0,1]$ has multiplicity 2 and $[1, -\frac{i}{\sqrt{3}}, -\frac{3}{4}]$ has multiplicity 1. We end up with

$$I_P(F, \mathrm{Hess}_F) = \begin{cases} 6 & \text{if } P = [0,0,1], \\ 1 & \text{if } P = [0,1,0], \\ 1 & \text{if } P = [1, \pm\frac{i}{\sqrt{3}}, -\frac{3}{4}], \\ 0 & \text{otherwise.} \end{cases}$$

The only singular point of C is $[0,0,1]$.

Exercise 18.

(a) Let $P \in C$. To obtain $-P$, we first have to look at the tangent line L of C at O and find its third intersection point R. Since O is a flex, the third intersection point is, again, O. The point $-P$ is then the third intersection point of the line through P and $R = O$. Consequently, $-P$ lies on the line through O, P. Let us prove that this statement implies

$$P + Q = O \;\Leftrightarrow\; P, Q, O \text{ on a line.}$$

If $P+Q = O$ then $Q = -P$ and $P, -P = Q, O$ lie on a line. If P, Q, O lie on a line then $Q = -P$ (since the third intersection point is unique). This implies that $P + Q = P + (-P) = O$.

(b) Let R be the third intersection point of the line through P, Q. Then $P + Q$ is the third intersection point of the line through R, O, by definition. Since $R, P+Q, O$ lie on a common line, we have $R = -(P+Q)$.

(c) Let P, Q, R lie on a common line. Then the third intersection point of the line through P, Q is $-(P+Q)$, by (b). Since the third intersection point is unique, we have $R = -(P+Q)$. Then $P+Q+R = P+Q-(P+Q) = O$, by group axioms. Conversely, let $P+Q+R = 0$. Then $R = -(P+Q)$, and R is the third intersection point of the line through P, Q, by (b). So P, Q, R lie on a common line.

Exercise 19.

(a) Assume that P has order three. Then $R = -2P$ is the third intersection point of the tangent of C at P. Since $3P = O$, the group operations yield $R = -2P + O = -2P + 3P = P$, so the tangent meets C at P with multiplicity 3. So P is a flex. Conversely, assume that $P \neq O$ is a flex. Then $R = -2P$ is the third intersection point of the tangent of C through P. Since P is a flex, this third point coincides with P. So we have $P = R = -2P$. This yields $3P = O$. P cannot be of order one since $P \neq O$. If it were of order two, i.e., $2P = O$, then we would obtain from $2P = O$ and $3P = O$ and the group operations that $P = O$, a contradiction. So the order of P is three.

(b) The third intersection point of the line through P, Q is $R = -(P+Q)$. Since P, Q are flexes, we have $3P = 3Q = 0$. So we conclude that $3R = -3(P+Q) = O$. So R is either of order 3 (and hence a flex) or $R = O$ (again, a flex) or of order 2. But $2R = 3R = O$ would yield, again $R = O$, so R would be also be a flex in this case.

Exercise 20. $F(X, Y, Z) = Z(X^2 + Y^2) - Y(3X^2 - Y^2)$ implies

$$\begin{aligned} F_X &= 2XZ - 6XY, \\ F_Y &= 2YZ - 3X^2 + 3Y^2, \\ F_Z &= X^2 + Y^2 = (X + iY)(X - iY). \end{aligned}$$

So $F_Z = 0$ leads to $X = \pm iY$ and $X^2 = -Y^2$. So $F_Y(\pm iY, Y, Z) = 2Y(Z - 6Y) = 0$ implies that $Y = 0$ or $Z = 6Y$. $X = Y = 0$ leads to the singularity $[0, 0, 1]$. The case $Z = 6Y$ leads to $F(\pm iY, Y, 6Y) = 4Y^3 = 0$, i.e., $Y = 0$, which implies $X = Z = 0$, a contradiction. So $[0, 0, 1]$ is the only singularity of C.

Next, we calculate the Hessian. We have

$$\begin{aligned} \mathrm{Hess}_F = \det \begin{pmatrix} 2Z - 6Y & -6Z & 2X \\ -6X & 2Z + 6Y & 2Y \\ 2X & 2Y & 0 \end{pmatrix} \\ = -48X^2Y - 4X^2(2Z + 6Y) = -8(Z(X^2 + Y^2) + 3Y(3X^2 - Y^2)) \\ = -8G(X, Y, Z). \end{aligned}$$

So $\mathrm{Hess}_F = F = 0$ leads to $Y(3X^2 - Y^2) = 0$, i.e., $Y = 0$ or $Y = \pm\sqrt{3}X$. The case $Y = 0$ leads to $ZX^2 = 0$, so $Z = 0$ or $X = 0$. So we end up with two points $[1, 0, 0]$, which is a flex, and with $[0, 0, 1]$ which is singular and, therefore, not a flex.

The case $Y = \pm\sqrt{3}X$ together with $Z(X^2 + Y^2) = 0$ leads to $4ZX^2 = 0$. So we conclude either $Z = 0$ and obtain the points $[1, \pm\sqrt{3}, 0]$, two flexes, or $X = 0$, which implies $Y = 0$ and leads, again, to the singularity $[0, 0, 1]$.

So the flexes of C are $[1,0,0]$, $[1,\sqrt{3},0]$ and $[1,-\sqrt{3},0]$. All three flexes lie on the line $Z=0$.

Exercise 21. Let $F(X,Y,Z)=Y^2Z+XYZ-X^3-XZ^2$. Then C is defined by F. We have

$$\begin{aligned}
F_X(X,Y,Z) &= YZ-3X^2-Z^2,\\
F_Y(X,Y,Z) &= 2YZ+XZ,\\
F_Z(X,Y,Z) &= Y^2+XY-2XZ.
\end{aligned}$$

$F_X(X,Y,Z)=0$ implies $Z=0$ or $X=-2Y$. $Z=0$ together with $F_Y(X,Y,Z)=0$ implies $X=0$, and $F_Z(X,Y,Z)=0$ finally yields $Y=0$. But $[0,0,0]$ is not a projective point. So a singular point has to be of the type $[-2Y,Y,1]$. We conclude from

$$F_Z(-2Y,Y,1)=-Y^2+4Y=-Y(Y-4)=0$$

that $Y=0$ or $Y=4$. But $F_Y(0,0,1)=-1$, so the only possible singularity is $[-8,4,1]$. But

$$F_X(-8,4,1)=4-3\cdot 64-1\neq 0.$$

So C is non-singular.

We have $\mathrm{Hess}_C=C_H$ with

$$H(X,Y,Z)=\det\begin{pmatrix}-6X & Z & Y-2Z\\ Z & 2Z & 2Y+X\\ Y-2Z & 2Y+X & -2X\end{pmatrix}.$$

The point $O=[0,1,0]$ is a flex if $F(0,1,0)=0$ and $H(0,1,0)=0$. $F(0,1,0)=0$ is obvious. Moreover, we have

$$H(0,1,0)=\det\begin{pmatrix}0&0&1\\0&0&2\\1&2&0\end{pmatrix}=0.$$

So, $O=[0,1,0]$ is a flex.

We first calculate $2P$: The tangent L_1 of C at P is given by

$$F_X(P)X+F_Y(P)Y+F_Z(P)Z=0,$$

i.e., $-3X+3Y=0$, so L_1 is given by $X=Y$. Next, we calculate the third point of the intersection $L_1 \cap C$:

$$F(X,X,Z) = 2X^2Z - X^3 - XZ^2 = -X(X-Z)^2.$$

So $F(X,X,Z)=0$ if and only if $X=Z$ (the point $[1,1,1]$ with multiplicity 2) or $X=0$. So the third point is $R=[0,0,1]$. The line L_2 through R and $O=[0,1,0]$ is given by $X=0$. Finally, we calculate the third point of the intersection $L_2 \cap C$:

$$F(0,Y,Z) = Y^2Z.$$

So $F(X,0,Y)=0$ if and only if $Z=0$ (the point O) or $Y=0$ (the point R with multiplicity 2). So the third point is, again, R. So $2P=R=[0,0,1] \neq O$. This implies $O=2R=4P$. The point P cannot be of order 3 since then the group operations would yield $P=O$, a contradiction.

Exercise 22. Two distinct projective lines $L_1 \cup L_2$ meet in precisely one point P. Let $C = L_1 \cup L_2 \subset \mathbb{P}^2$. Any open neighbourhood $V \subset C$ of P is therefore the intersection of an open set $U \subset \mathbb{P}^2$ with C, i.e., $V = U \cap C$. Removing the point P from V disconnects the set V, i.e., when choosing $Q_1 \in (L_1 \cap V) - \{P\}$ and $Q_2 \in (L_2 \cap V) - \{P\}$, then there is no continuous path in $V - \{P\}$, connecting Q_1 with Q_2 (since such a path would have to pass through the only common point of L_1 and L_2, namely P). Assume that C is a Riemann surface. Choosing V small enough and path connected, V has then to be homeomorphic to a path connected open set $\tilde{V}$ of $\mathbb{C}$. But $\tilde{V}$ remains obviously path connected when removing a single point. This is a contradiction!

Exercise 23. (a) A non-singular cubic has even 9 different flexes, all of order one, by a corollary of the course. Choosing a flex P, we can assume, after coordinate change, that $P=[0,1,0]$ and that the tangent line of C at P is $Z=0$. By arguments of the course, we then have

$$F(X,Y,Z) = ZG(X,Y,Z) + X^3H(X,Z)$$

with $G(0,1,0) \neq 0 \neq H(0,1)$. Since F has degree three, H has degree zero and is, therefore, a non-zero constant a (because of $H(0,1) \neq 0$). G has degree 2 with non-vanishing term Y^2, since $G(0,1,0) \neq 0$:

$$G(X,Y,Z) = bX^2 + cXY + dXZ + eY^2 + fYZ + gZ^2,$$

with $e \neq 0$. So

$$F(X,Y,Z) = aX^3 + bX^2Z + cXYZ + dXZ^2 + eY^2Z + fYZ^2 + gZ^3.$$

(b) The substitution yields

$$\begin{aligned}
&aX^3 + bX^2Z + cX(Y - \frac{c}{2e}X - \frac{f}{2e}Z)Z + dXZ^2 + e(Y - \frac{c}{2e}X - \frac{f}{2e}Z)^2Z \\
&+ f(Y - \frac{c}{2e}X - \frac{f}{2e}Z)Z^2 + gZ^3 = aX^3 + bX^2Z + cXYZ - \frac{c^2}{2e}X^2Z - \frac{cf}{2e}XZ^2 \\
&+ dXZ^2 + eY^2Z + \frac{c^2}{4e}X^2Z + \frac{f^2}{4e}Z^3 - cXYZ - fYZ^2 + \frac{cf}{2e}XZ^2 + fYZ^2 - \frac{cf}{2e}XZ^2 \\
&\qquad - \frac{f^2}{2e}Z^3 + gZ^3 = aX^3 + b'X^2Z + d'XZ^2 + eY^2Z + g'Z^3,
\end{aligned}$$

with the same non-zero coefficients a and e of X^3 and Y^2Z.

(c) The substitution yields

$$\begin{aligned}
&a'(X - \frac{b'}{3a'}Z)^3 + b'(X - \frac{b'}{3a'}Z)^2Z + d'(X - \frac{b'}{3a'}Z))Z^2 + e'Y^2Z + g'Z^3 \\
&\qquad = a'X^3 + (-3a'\frac{b'}{3a'} - b')X^2Z + d''XZ^2 + e'Y^2Z + g''Z^3 \\
&\qquad\qquad = a'X^3 + d''XZ^2 + e'Y^2Z + g''Z^3,
\end{aligned}$$

with the same non-zero coefficients a' and e' of X^3 and Y^2Z.

(d) Since $e' \neq 0$, we can arrange $e' = 1$ by substituting Z by $\frac{1}{e'}Z$. Since $a' \neq 0$, we can arrange $a' = -4$ by rescaling X appropriately. So we end up with

$$Y^2Z = 4X^3 - d''XZ^2 - g''Z^3.$$

Choosing $\alpha = d''$ and $\beta = g''$, we obtain the desired *Weierstraß normal form.*