Prof. Norbert Peyerimhoff, Durham University 29/1/2014

# Algebraic Geometry III/IV

**Problems, set 2.** To be handed in on **Wednesday, 5 February 2014**, in the lecture.

**Exercise 3.** Let $C_F = \{[a,b,c] \mid F(a,b,c) = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a projective curve and $P \in C_F$. Let $L \subset \mathbb{P}^2_{\mathbb{C}}$ be a line through $P$. Let $f,g,h : (-T,T) \to \mathbb{C}$ be differentiable functions such that

$$[f(t), g(t), h(t)] \in L \qquad \forall t \in (-T,T)$$

and $P = [f(0), g(0), h(0)]$, and let $k : (-T,T) \to \mathbb{C}$ be defined by

$$k(t) = F(f(t), g(t), h(t)).$$

Show the following fact: If $k'(0) \neq 0$ then $P$ is a nonsingular point of $C_F$ and $L$ is not the tangent line of $C_F$ at the point $P$.

**Exercise 4.** This exercise leads you through the proof that every nonsingular projective cubic $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ has precisely 9 different flexes. (In last term's course you saw a proof that $C_F$ has at least one flex.)

So assume that $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ is a nonsingular projective cubic and

$$\mathcal{H}_F = \det \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.$$

(a) Show first that $C_F \cap C_{\mathcal{H}_F}$ is finite and that

$$\sum_{P \in C_F \cap C_{\mathcal{H}_F}} \mathrm{ind}_P(F, \mathcal{H}_F) = 9.$$

(b) Let $P \in C_F$ be a flex. From last term's lecture we can assume without loss of generality that $P = [0,1,0]$ and that $F$ has the form

$$F(X,Y,Z) = Y^2 Z - X(X-Z)(X-\lambda Z)$$

with $\lambda \in \mathbb{C} - \{0,1\}$ (via a suitable projective transformation). Check that the tangent line $L$ of $C_F$ at $P$ is given by $Z = 0$.

(c) The points $P_t = [t, 1, 0]$ lie in the tangent line $L : Z = 0$ given in (b), for all $t \in \mathbb{R}$. Calculate $k(t) = \mathcal{H}_F(P_t)$ explicitely.

(d) Conclude with the help of Exercise 3 that $P$ is a nonsingular point of the curve $\mathcal{H}_F = 0$ and that the line $L : Z = 0$ cannot be its tangent line.

(e) Use the previous results to show that $C_F$ has precisely 9 flexes.