Prof. Norbert Peyerimhoff, Durham University 5/2/2014

Algebraic Geometry III/IV

Problems, set 3.

Exercise 5. (Zariski Topology) Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is called a *closed subset* iff its complement $A^c = X \backslash A$ is open, i.e.,

$$A \text{ closed} \Leftrightarrow A^c \in \mathcal{T}.$$

Let $\mathcal{C}$ be the collection of all closed subsets of X. Obviously, a set X can also be made to a topological space by specifying all its closed sets, i.e., by giving $\mathcal{C}$. Check first that the conditions on the open sets then translate into

(a) $\emptyset, X \in \mathcal{C}$.

(b) If $A_1, \dots, A_k$ are finitely many closed sets (i.e., $A_j \in \mathcal{C}$ for $1 \le j \le k$) then we also have $\bigcup_{j=1}^k A_j \in \mathcal{C}$.

(c) If $(A_\alpha)_{\alpha \in I}$ is an arbitrary (finite or infinite) collection of closed sets (i.e., $A_\alpha \in \mathcal{C}$ for all $\alpha \in I$) then we also have $\bigcap_{\alpha \in I} A_\alpha \in \mathcal{C}$.

Now we introduce algebraic subsets of $\mathbb{C}^2$. Let $f(x,y) \in \mathbb{C}[x,y]$ be a polynomial. The subset of points $(a,b) \in \mathbb{C}^2$ with $f(a,b) = 0$ is denoted by V_f ("variety of F"). A subset $A \subset \mathbb{C}^2$ is called an *algebraic set*, if there are finitely many polynomials $f_1, \dots, f_k$ ($k = 0$ is allowed) such that $A = V_{f_1} \cap \dots \cap V_{f_k}$. In the *Zariski topology* of $\mathbb{C}^2$, the closed subsets are precisely the algebraic sets. Show that the Zariski topology makes indeed $\mathbb{C}^2$ a topological space. You may use the following fact without proof: Every ideal $I \subset \mathbb{C}[x,y]$ is finitely generated, i.e., there are finitely many polynomials $f_1, \dots, f_k \in \mathbb{C}[x,y]$ such that

$$I = \{\sum_{i=1}^k f_i g_i \mid g_i \in \mathbb{C}[x,y]\}.$$

(This is an application of Hilbert's Basis Theorem.) Finally show that every open set $U \subset \mathbb{C}^2$ in the Zariski topology is also open in the usual Euclidean topology of $\mathbb{C}^2$. You may use the fact that if $\mathbb{C}, \mathbb{C}^2$ carry the usual Euclidean topology, then every polynomial $f(x,y) \in \mathbb{C}[x,y]$ defines a continuous function $f : \mathbb{C}^2 \to \mathbb{C}$.