Prof. Norbert Peyerimhoff, Durham University 22/1/2014

Algebraic Geometry III/IV

Solutions, set 1.

Exercise 1. Let $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \in \mathbb{C}^3$ be lifts of of P_0, P_1, P_2, P_3, i.e., if $P_i = [a, b, c]$ then $tildeP_i = \lambda(a, b, c)$ for some chosen $\lambda \in \mathbb{C} - 0$. If $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$ would lie in a hyperplane of $\mathbb{C}^3$ through the origin, then P_0, P_1, P_2 would lie on a common projective line, which is not allowed. Therefore, $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$ are linearly independent and we can find a linear isomorphism $\alpha : \mathbb{C}^3 \to \mathbb{C}^3$ such that $\alpha(\tilde{P}_1) = (1, 0, 0)$, $\alpha(\tilde{P}_2) = (0, 1, 0)$ and $\alpha(\tilde{P}_3) = (0, 0, 1)$. Let $\alpha(\tilde{P}_4) = (a, b, c) \in \mathbb{C}^3$. Note that none of the numbers a, b, c can be equal to 0: For example, if $b = 0$, the points $(1, 0, 0), (0, 0, 1), (a, b, c)$ would lie in a hyperplane of $\mathbb{C}^3$ through the origin, and so would the points $\tilde{P}_1, \tilde{P}_3, \tilde{P}_4$ and, therefore, P_1, P_3, P_4 would lie on a common projective line, which is not allowed. Choose $\beta : \mathbb{C}^3 \to \mathbb{C}^3$ the unique linear isomorphism such that $\beta(1, 0, 0) = (1/a, 0, 0)$, $\beta(0, 1, 0) = (0, 1/b, 0)$ and $\beta(0, 0, 1) = (0, 0, 1/c)$. Then the linear isomorphism $\gamma = \beta \circ \alpha : \mathbb{C}^3 \to \mathbb{C}^3$ satisfies $\gamma(\tilde{P}_1) = (1/a, 0, 0)$, $\gamma(\tilde{P}_2) = (0, 1/b, 0)$ and $\gamma(\tilde{P}_3) = (0, 0, 1/c)$, $\gamma(\tilde{P}_4) = (1, 1, 1)$. The corresponding projective transformation $f : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ satisfies then $f(P_0) = [1, 0, 0]$, $f(P_1) = [0, 1, 0]$, $f(P_2) = [0, 0, 1]$ and $f(P_3) = [1, 1, 1]$.

Exercise 2. Let $P_1, \dots, P_5$ be five different points in $\mathbb{P}^2_{\mathbb{C}}$.

(a) We first assume that no three of these points lie on a common projective line. Using Exercise 1, we can assume without loss of generality that $P_1 = [1, 0, 0], P_2 = [0, 1, 0], P_3 = [0, 0, 1], P_4 = [1, 1, 1]$ (since we can establish this situation via a projective transformation). Then we also have $P_5 = [1, a, b]$, since if $P_5 = [0, a, b]$ the points P_2, P_3, P_5 would lie on a common projective line, which is not allowed. Similarly, we cannot have $a = 0$ or $b = 0$ or$a = 1$ or $b = 1$ or $a = b$. (For example, if $a = b$, the three points P_0, P_4, P_5 wouldlie on a common projective line. Any conic C must have the form $AX^2 + BY^2 + CZ^2 + DXY + EXZ + FYZ = 0$, where at least one of the coefficients $A, \dots, F$ is not zero. The conditions $P_1, P_2, P_3 \in C$ imply that $A = B = C = 0$, i.e., the conic has the form $DXY + EXZ + FYZ = 0$. The condition $P_4 \in C$

implies that $D+E+F=0$. Finally, the condition $P_5 \in C$ leads to $aD+bE+abF=0$. Solving the simultaneous equations $D+E+F=0$ and $aD+bE+abF=0$ leads to the equation $b(1-a)XY-a(1-b)XZ+(a-b)YZ=0$, which is unique up to multiplication by a non-zero complex constant. Note also that all three coefficients $b(1-a)$, $-a(1-b)$ and $a-b$ are non-zero. This unique conic containing P_1, P_2, P_3, P_4 cannot be reducible since, otherwise, it would consist of the union of two projective lines and one of the these lines would have to contain at least three of the five points, which is not allowed.

(b) If only P_1, P_2, P_3 lie on a common projective line L and P_4, P_5 do not lie on L, then we can find a unique projective line $L' \neq L$ containing P_4, P_5. Then any conic C containing P_1, P_2, P_3 must also contain L, since the intersection of a conic C with a line L are at most two points, unless L is a component of C. Such a conic C must be reducible and, therefore, consist of two lines, one of them L. Since $P_4, P_5 \notin L$, P_4, P_5 must lie on the other line of C and there is only one line containing P_4, P_5, namely L', and we have $C = L \cup L'$. This shows that C is unique and reducible.

(c) If $P_1, \ldots, P_4$ lie on a common projective line L, then L must be part of any conic C containing all five points, and therefore C must be reducible with the only condition that P_5 lies on the other line L' of C, if $P_5 \notin L$. (If $P_5 \in L$, there is no further condition on L'.) Obviously, there are infinitely many projective lines L' satisfying $P_5 \in L'$ (or without any condition) and all the corresponding conics $L \cup L'$ are reducible.