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Algebraic Geometry III/IV

Solutions, set 2.

Exercise 3. Using the chain rule when differentiating $k(t)$ we obtain

$$k'(t) = F_X(f(t), g(t), h(t))f'(t) + F_Y(f(t), g(t), h(t))g'(t) \\ + F_Z(f(t), g(t), h(t))h'(t).$$

Setting $t = 0$ and recalling that $P = [f(0), g(0), h(0)]$ we conclude that

$$k'(0) = F_X(P)f'(0) + F_Y(P)g'(0) + F_Z(P)h'(0).$$

Now, if $k'(0) \neq 0$, then at least one of $F_X(P), F_Y(P), F_Z(P)$ is not vanishing and, therefore, P is a nonsingular point of C_F. Recall that the tangent line L' of C_F at the nonsingular point P is given by

$$F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0.$$

If we would have $L' = L$ then we could conclude $[f(t), g(t), h(t)] \in L'$ for all $t \in (-T, T)$ and, therefore, by differentiation

$$0 = \frac{d}{dt}\Big|_{t=0} \underbrace{(F_X(P)f(t) + F_Y(P)g(t) + F_Z(P)h(t))}_{=0} \\ = F_X(P)f'(0) + F_Y(P)g'(0) + F_Z(P)h'(0) = k'(0).$$

But this would be in contradiction to $k'(0) \neq 0$.

Exercise 4. Let $C_F \subset \mathbb{P}^2_{\mathbb{C}}$ be a nonsingular projective cubic and

$$\mathcal{H}_F = \det \begin{pmatrix} F_{XX} & F_{XZ} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}$$

be its Hessian.

(a) First of all, C_F and $C_{\mathcal{H}_F}$ do not share a common factor for, otherwise, we would have $C_F \subset C_{\mathcal{H}_F}$, because C_F is nonsingular and, therefore, irreducible. But then every point of C_F is a flex. A theorem in last term's lecture states that this implies that $\deg C_F = 1$, which is a contradiction. Recall that $\deg \mathcal{H}_G = 3(d-2)$ where d is the degree of G. Since $\deg F = 3$, we have $\deg \mathcal{H}_G = 3$, and we can now apply Bezout's Theorem and conclude that $C_F \cap C_{\mathcal{H}_F}$ is finite and

$$\sum_{P \in C_F \cap C_{\mathcal{H}_F}} \mathrm{ind}_P(F, \mathcal{H}_F) = 3 \cdot 3 = 9.$$

(b) Using the comments in the exercise, we can assume that $P = [0, 1, 0]$ and

$$F(X, Y, Z) = Y^2 Z - X^3 + (1+\lambda) X^2 Z - \lambda X Z^2$$

for some $\lambda \in \mathbb{C} - \{0, 1\}$. Then we have

$$\begin{aligned} F_X &= -3X^2 + 2(1+\lambda)XZ - \lambda Z^2, \\ F_Y &= 2YZ, \\ F_Z &= Y^2 + (1+\lambda)X^2 - 2\lambda XZ, \end{aligned}$$

and the tangent line L is given by the equation

$$F_X(0,1,0)X + F_Y(0,1,0)Y + F_Z(0,1,0)Z = 0.$$

The statement follows now from $F_X(0,1,0) = 0, F_Y(0,1,0) = 1$ and $F_Z(0,1,0) = 1$.

(c) We have

$$F_{XX} = -6X + 2(1+\lambda)Z, \quad F_{XY} = 0, \quad F_{XZ} = 2(1+\lambda)X - 2\lambda Z$$

and

$$F_{YX} = 0, \quad , F_{YY} = 2Z, \quad F_{YZ} = 2Y,$$

and

$$F_{ZX} = 2(1+\lambda)X - 2\lambda Z, \quad F_{ZY} = 2Y, \quad F_{ZZ} = -2\lambda X,$$

and therefore

$$k(t) = \mathcal{H}_F(t, 1, 0) = \det \begin{pmatrix} -6t & 0 & 2(1+\lambda)t \\ 0 & 0 & 2 \\ 2(1+\lambda)t & 2 & -2\lambda t \end{pmatrix} = 24t.$$

(d) Choosing $f(t) = t, g(t) = 1, h(t) = 0$, we have

$$k(t) = \mathcal{H}_F(f(t), g(t), h(t)) = 24t$$

and $k'(0) = 24 \neq 0$. Moreover, if L denotes the line $Z = 0$, which is the tangent line of C_F at the nonsingular point $P = [0, 1, 0]$, then $[f(t), g(t), h(t)] \in L$ for all $t \in \mathbb{R}$, and we can apply Exercise 3 to conclude that P is also a nonsingular point of the curve $\mathcal{H}_F = 0$ and that L is not the tangent line of $\mathcal{H}_F = 0$ at P.

(e) Recall that we started with an arbitrary flex P of C_F. The result in (d) shows that

$$\mathrm{ind}_P(F, \mathcal{H}_F) = 1.$$

Therefore all intersection indices in the formula in (a) are equal to 1 and there must be nine summands. But the flexes of C_F agree precisely with the points of the intersection $C_F \cap C_{\mathcal{H}_F}$, so the curve C_F must have precisely 9 flexes.