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Algebraic Geometry III/IV

Solutions, set 3.

Exercise 5. Since $\emptyset^c = X$ and $X^c = \emptyset$, observe first that the conditions $\emptyset, X \in \mathcal{T}$ and $\emptyset, X \in \mathcal{C}$ are equivalent. Recall that $A \in \mathcal{T}$ if and only if $A^c \in \mathcal{C}$. Therefore, the remaining conditions for the closed sets follow from *de Morgan's Rule* for finite and infinite unions and intersections, namely,

$$\begin{aligned} \left(\bigcup_{j=1}^k A_j \right)^c &= \bigcap_{j=1}^k A_j^c, \\ \left(\bigcap_{\alpha \in I} A_\alpha \right)^c &= \bigcup_{\alpha \in I} A_\alpha^c. \end{aligned}$$

We need to check (a), (b), (c) for the algebraic sets. First of all, X is the empty intersection of varieties, and we also have $\emptyset = V_f$ for $f(x,y) = 1$. This shows that (a) is satisfied. (b) is also relatively easy: Let $A_1, \dots, A_k$ be algebraic sets, given by $A_j = V_{f_{j,1}} \cap \dots \cap V_{f_{j,n}}$. (We can choose the same number n for all sets A_j, since we can always add trivial polynomials $f_{j,l}(x,y) = 0$ without changing anything). Then we obviously have

$$A := \bigcup_{j=1}^k A_j = \bigcap_{1 \le i_1, \dots, i_k \le n} V_{f_{1,i_1} \cdots f_{k,i_k}},$$

i.e., A is also an algebraic set.

It remains to prove (c). Let $A = \bigcap_{\alpha \in I} A_\alpha$. Since each A_α is an intersection of finitely varieties V_f, we can assume, without loss of generality, that $A_\alpha = V_{f_\alpha}$ (by renaming the sets involved in $\bigcap_{\alpha \in I} A_\alpha$). Let I be the ideal generated by all polynomials f_α, i.e.,

$$I = \left\{ \sum_{\alpha \in I} f_\alpha g_\alpha \mid \text{only finitely many } g_\alpha \in \mathbb{C}[x,y] \text{ nonzero} \right\}.$$

It is easy to see that I is an ideal, since we have for two polynomials $f_1, f_2 \in I$ and $g \in \mathbb{C}[x, y]$,

$$f_1 + f_2 \in I, \qquad f_1 g \in I.$$

We know from the exercise sheet that there are finitely many polynomials $f_1, \ldots, f_k$ such that

$$I = \left\{ \sum_{j=1}^{k} f_j g_j \mid g_j \in \mathbb{C}[x, y] \right\}.$$

Our goal is to show that

$$A = V_{f_1} \cap \cdots \cap V_{f_k},$$

then A is an algebraic set and we proved (c). First of all, choose $(a, b) \in A = \bigcap_{\alpha \in I} V_{f_\alpha}$. Then we know that $f_\alpha(a, b) = 0$ for all $\alpha \in I$. We also know for all $i \in \{1, 2, \ldots, k\}$ that $f_i \in I$ and, therefore, we can write

$$f_i = \sum_{\alpha \in I} f_\alpha g_\alpha$$

with only finitely many g_α nonzero. This implies that

$$f_i(a, b) = \sum_{\alpha \in I} \underbrace{f_\alpha(a, b)}_{=0} g_\alpha(a, b) = 0,$$

i.e., $(a, b) \in V_{f_i}$. Since this argument works for every $i \in \{1, 2, \ldots, k\}$, we see that $(a, b) \in V_{f_1} \cap \cdots \cap V_{f_k}$. This shows one of the two inclusions. Conversely, choose now $(a, b) \in V_{f_1} \cap \cdots \cap V_{f_k}$. We need to show that $f_\alpha(a, b) = 0$ for all $\alpha \in I$, since then $(a, b) \in A = \bigcap_{\alpha \in I} V_{f_\alpha}$. But every polynomial f_α lies in the ideal I and can, therefore, be written as

$$f_\alpha = \sum_{j=1}^{k} f_j g_j$$

with suitable polynomials $g_j \in \mathbb{C}[x, y]$. This implies that

$$f_\alpha(a, b) = \sum_{j=1}^{k} \underbrace{f_j(a, b)}_{=0} g_j(a, b) = 0,$$

finishing the proof of the reverse inclusion.

Finally, let $U \subset \mathbb{C}^2$ be an open set in the Zariski topology. This means that its complement U^c is an algebraic set and, therefore, U^c is the intersection of finitely varieties, i.e., $U^c = V_{f_1} \cap \cdots \cap V_{f_k}$ with suitable polynomials $f_1, \ldots, f_k \in \mathbb{C}[x, y]$. We now explain the fact that the varieties $V_{f_i} \subset \mathbb{C}^2$ are also closed sets in the usual Euclidean topology: $\mathbb{C}\backslash 0$ is obviously an open subset of $\mathbb{C}$ with respect to the usual Euclidean topology (write $\mathbb{C}\backslash 0 = \bigcap_{z \neq 0} B_{|z|}(z)$) and f_i is continuous. Therefore the preimage $(V_{f_i})^c = f_i^{-1}(\mathbb{C}\backslash 0)$ of $\mathbb{C}\backslash 0$ under f_i is open in $\mathbb{C}^2$. The intersection of the finitely many closed sets V_{f_i} with respect to the usual Euclidean topology is again closed in this topology. Therefore, its complement $U = (V_{f_1} \cap \cdots \cap V_{f_k})^c$ is open with respect to the usual Euclidean topology.