Prof. Norbert Peyerimhoff, Durham University 5/3/2014

Algebraic Geometry III/IV

Solutions, set 7.

Exercise 10.

(a) Let $f(x,y) = x^2 - x^4 - y^4$. We have

$$\begin{aligned} f_x(x,y) &= 2x(1-\sqrt{2}x)(1+\sqrt{2}x), \\ f_y(x,y) &= -4y^3. \end{aligned}$$

The condition $f_y(x,y) = 0$ implies $y = 0$ and, because of $f(x,0) = x^2(1-x)(1+x)$, we see that $f(x,y) = f_x(x,y) = f_y(x,y) = 0$ has the only solution $(x,y) = (0,0)$. Next we calculate the tangent lines of C_f at $(0,0)$. We have

$$f(x,y) = f_2(x,y) + f_4(x,y),$$

with $f_2(x,y) = x^2$ and $f_4(x,y) = -x^4 - y^4$. Therefore, we have a double tangent line given by $x = 0$. So we need to consider the blow-up in U_1. We set $(x,y) = (x_1y_1, y_1)$ and obtain

$$f(x_1y_1, y_1) = y_1^2(x_1^2 - x_1^4y_1^2 - y_1^2),$$

so the strict transform of f in U_1 is

$$f^{(1)}(x_1,y_1) = x_1^2 - x_1^4y_1^2 - y_1^2.$$

We now have

$$\begin{aligned} f^{(1)}_{x_1}(x_1,y_1) &= 2x_1(1-2x_1^2y_1^2), \\ f^{(1)}_{y_1}(x_1,y_1) &= -2y_1(x_1^4+1). \end{aligned}$$

The preimages of $(x,y) = (0,0)$ under the strict transform are given by $y_1 = y = 0$ and $f^{(1)}(x_1,0) = x_1^2 = 0$, i.e., only $(x_1,y_1) = (0,0)$, which is still a singular point of $C_{f^{(1)}}$. Next we calculate the tangent lines of $C_{f^{(1)}}$ at $(0,0)$. We have

$$f^{(1)}(x_1, y_1) = f_2^{(1)}(x_1, y_1) + f_6^{(1)}(x_1, y_1)$$

with $f_2^{(1)}(x_1, y_1) = (x_1 - y_1)(x_1 + y_1)$ and $f_6^{(1)}(x_1, y_1) = -x_1^4 y_1^2$. Therefore, we have the two tangent lines $x_1 = y_1$ and $x_1 = -y_1$. So we can consider the next blow-up in U_0. We set $(x_1, y_1) = (x_2, x_2 y_2)$ and obtain

$$f^{(1)}(x_2, x_2 y_2) = x_2^2(1 - x_2^4 y_2^2 - y_2^2),$$

so the strict transform of $f^{(1)}$ in U_0 is

$$f^{(2)}(x_2, y_2) = 1 - x_2^4 y_2^2 - y_2^2.$$

The preimages of $(x_1, y_1) = (0,0)$ under the strict transform are given by $x_2 = x_1 = 0$ and $f^{(2)}(0, y_2) = 1 - y_2^2 = 0$, i.e., $(x_2, y_2) = (0, \pm 1)$. We now have

$$\begin{aligned} f_{x_2}^{(2)}(x_2, y_2) &= -4x_2^3 y_2^2, \\ f_{y_2}^{(2)}(x_2, y_2) &= -2y_2(x_2^4 + 1). \end{aligned}$$

Since $f_{y_2}^{(2)}(0, \pm 1) = \mp 2 \neq 0$, all singularities are now resolved and the blow-up process stops.

(b) Let $g(x, y) = y^3 - x^5$. We have

$$\begin{aligned} g_x(x, y) &= -5x^4, \\ g_y(x, y) &= 3y^2, \end{aligned}$$

so $g_x(x, y) = g_y(x, y) = 0$ implies that $(x, y) = (0, 0)$. Therefore, the only singularity of C_g is $(0,0)$. The tangent lines of C_g at $(0,0)$ are $y = 0$ (trice) and we can consider the blow-up in U_0. We set $(x, y) = (x_1, x_1 y_1)$ and obtain

$$g(x_1, x_1 y_1) = x_1^3(y_1^3 - x_1^2),$$

i.e., the strict transform of g in U_0 is

$$g^{(1)}(x_1, y_1) = y_1^3 - x_1^2.$$

The preimages of $(x, y) = (0, 0)$ under the strict transform are given by $x_1 = x = 0$ and $g^{(1)}(0, y_1) = y_1^3 = 0$, i.e., $(x_1, y_1) = (0, 0)$. Since

$$
\begin{aligned}
g^{(1)}_{x_1}(x_1, y_1) &= -2x_1, \\
g^{(1)}_{y_1}(x_1, y_1) &= -3y_1^2,
\end{aligned}
$$

the point $(0, 0)$ is still a singularity of $C_{g^{(1)}}$. The tangent lines of $C_{g^{(1)}}$ at $(0, 0)$ are $x = 0$ (twice) and we need to carry out the blow-up in U_1. We set $(x_1, y_1) = (x_2y_2, y_2)$ and obtain

$$g^{(1)}(x_2y_2, y_2) = y_2^2(y_2 - x_2^2).$$

The strict transform of $g^{(1)}$ in U_1 is therefore

$$g^{(2)}(x_2, y_2) = y_2 - x_2^2.$$

The preimages of $(x_1, y_1) = (0, 0)$ under the strict transform are given by $y_2 = y_1 = 0$ and $g^{(2)}(x_2, 0) = -x_2^2 = 0$, i.e., $(x_2, y_2) = (0, 0)$. Since $g^{(2)}_{y_2}(x_2, y_2) = 1 \neq 0$, all singularities are resolved and the blow-up process stops.