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Algebraic Geometry III/IV

Solutions, set 8.

Exercise 11. Recall that $F(X,Y,Z) = Y^5 - X^5 + X^2Z^3$.

(a) We have $F(0,1,0) = 1^5 - 0^5 + 0^5 = 1 \neq 0$, i.e., $[0,1,0] \notin C_F$. This guarantees that the map $\pi : C_F \to \mathbb{P}^1_{\mathbb{C}}$, $\pi([a,b,c]) = [a,c]$ is well defined.

(b) We have $F_Y(X,Y,Z) = 5Y^4$. So the solutions of $F(P) = F_Y(P) = 0$ are given by $Y = 0$ and $F(X,0,Z) = X^2(Z^3 - X^3)$, i.e., $P_1 = [0,0,1]$ and $P_2 = [1,0,1], P_3 = [1,0,\zeta_3], P_4 = [1,0,\zeta_3^2]$ with $\zeta_3 = e^{2\pi i/3}$. Thus we have $R = \{P_1, P_2, P_3, P_4\}$ and

$$B = \pi(R) = \{[0,1],[1,1],[1,\zeta_3],[1,\zeta_3^2]\}.$$

We see that B contains 4 points.

(c) Since the singularities are a subset of R, we only need to check for which of these points $P \in R$ we have $F_X(P) = F_Z(P) = 0$. We have

$$F_X(X,Y,Z) = -5X^4 + 2XZ^3, \qquad , F_Z(X,Y,Z) = 3X^2Z^2.$$

Since $F_Z(1,0,\zeta_3^i) = 3\zeta_3^{2i} \neq 0$, we see that $P_2, P_3, P_4 \notin \mathrm{Sing}(C_F)$. We conclude that $\mathrm{Sing}(C_F) = \{P_1\} \subset R$, since

$$F_X(0,0,1) = 0, \qquad F_Z(0,0,1) = 0.$$

(d) We only need to carry out the blow-up procedure in the singular point P_1. We first choose affine coordinates via the identification $(x,y) \mapsto [x,y,1]$ and obtain the affine polynomial

$$f(x,y) = F(x,y,1) = x^2 + y^5 - x^5.$$

We see that we have a double tangent line given by $x = 0$. So we need to consider the blow-up in U_1. We set $(x,y) = (x_1y_1, y_1)$ and obtain

$$f(x_1y_1, y_1) = y_1^2(x_1^2 + y_1^3 - x_1^5y_1^3),$$

so the strict transform of f in U_1 is

$$f^{(1)}(x_1, y_1) = x_1^2 + y_1^3 - x_1^5 y_1^3.$$

The preimages of $(x, y) = (0, 0)$ under the strict transform are given by $y_1 = y = 0$ and $f^{(1)}(x_1, 0) = x_1^2 = 0$, i.e., only $(x_1, y_1) = (0, 0)$, which is still a singular point of $C_{f^{(1)}}$ since

$$f^{(1)}_{x_1}(x_1, y_1) = 2x_1 - 5x_1^4 y_1^3, \qquad f^{(1)}_{y_1}(x_1, y_1) = 3y_1^2 - 3x_1^5 y_1^2.$$

At $(0, 0) \in C_{f^{(1)}}$, we have again a double tangent line given by $x_1 = 0$. So we need to carry out the next blow-up again in U_1. We obtain

$$f^{(1)}(x_2 y_2, y_2) = y_2^2(y_2 + x_2^2 - x_2^5 y_2^6),$$

so the strict transform of $f^{(1)}$ in U_1 is

$$f^{(2)}(x_2, y_2) = y_2 + x_2^2 - x_2^5 y_2^6.$$

The preimages of $(x_1, y_1) = (0, 0)$ under the strict transform are given by $y_2 = y_1 = 0$ and $f^{(2)}(x_2, 0) = x_2^2 = 0$, i.e., only $(x_2, y_2) = (0, 0)$. Since $f^{(2)}_{y_2}(x_2, y_2) = 1 \neq 0$, the point $(0, 0) \in C_{f^{(2)}}$ is no longer singular and the blow-up process stops with a non-singular model $\psi : \widetilde{C} \to C_F$.

(e) Since B contains the 4 points $P_1, P_2, P_3, P_4 \in \mathbb{P}^1_{\mathbb{C}}$, we know from a result in the lectures that there exists a triangulation $\mathcal{T}$ of $\mathbb{P}^1_{\mathbb{C}}$ with the four vertices P_1, P_2, P_3, P_4 and $3 \cdot 4 - 6 = 6$ edges and $2 \cdot 4 - 4 = 4$ triangles. The preimage $R = \pi^{-1}(B) \subset C_F$ contains also 4 points, and the preimage of R under the blow-up procedure $\psi : \widetilde{C} \to C_F$ consists again of only four different points. Since $\deg F = 5$, we end up with an induced triangulation of $\widetilde{C}$ with $V = 4$ vertices, $E = 5 \cdot 6 = 30$ edges and $F = 5 \cdot 4 = 20$ triangles. This implies that $\widetilde{C}$ has the Euler number

$$\chi(\widetilde{C}) = V - E + F = 4 - 30 + 20 = -6.$$

(f) Using the relation $\chi(\widetilde{C}) = 2 - 2g(\widetilde{C})$, we conclude that the genus of the non-singular model $\widetilde{C}$ is

$$g(\widetilde{C}) = 1 - \frac{\chi(\widetilde{C})}{2} = 1 - \frac{-6}{2} = 4.$$