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Algebraic Geometry III/IV

Solutions, set 9.

Exercise 12.

(a) The polynomial $F(X,Y,Z)$ is given by

$$F(X,Y,Z) = aX^2 + bY^2 + cZ^2 + 2dXY + 2eXZ + 2fYZ.$$

It is easy to see that the condition $(x,y,z) \neq 0$ and $F_X(x,y,z) = F_Y(x,y,z) = F_Z(x,y,z) = 0$ is equivalent to

$$\begin{aligned} 2ax + 2dy + 2ez &= 0, \\ 2by + 2dx + 2fz &= 0, \\ 2cz + 2ex + 2fy &= 0, \end{aligned}$$

which, in turn, is equivalent to

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

This means, we have a nontrivial simultaneous solution $F_X(x,y,z) = F_Y(x,y,z) = F_Z(x,y,z) = 0$ if and only if A has a nontrivial kernel, i.e., if and only if $\det A = 0$. But any such solution satisfies obviously also $F(x,y,z) = 0$, i.e., is a singular point of C_F, and vice versa.

(b) The tangent line of C_F at $[\alpha,\beta,\gamma] \in C_F$ is given by the equation

$$F_X(\alpha,\beta,\gamma)X + F_Y(\alpha,\beta,\gamma)Y + F_Z(\alpha,\beta,\gamma)Z = 0,$$

i.e.,

$$(2a\alpha + 2d\beta + 2e\gamma)X + (2b\beta + 2d\alpha + 2f\gamma)Y + (2c\gamma + 2e\alpha + 2f\beta)Z = 0,$$

i.e.,

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0.$$

(c) We conclude from (b) that

$$\mathcal{T}(C) = \left\{ C_H \mid H(X,Y,Z) = \begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix} A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \text{ and } [\alpha,\beta,\gamma] \in C \right\}.$$

This implies that

$$C^* = \Phi(\mathcal{T}(C)) = \{[x,y,z] \in \mathbb{P}^2_{\mathbb{C}} \mid (x\ y\ z) = (\alpha\ \beta\ \gamma)A \text{ for some } [\alpha,\beta,\gamma] \in C\},$$

i.e.,

$$C^* = \{[x,y,z] \in \mathbb{P}^2_{\mathbb{C}} \mid [(x\ y\ z)A^{-1}] \in C\}.$$

Now we have for every $(x\ y\ z) \neq 0$, using $A^\top = A$,

$$\begin{aligned} [(x\ y\ z)A^{-1}] \in C \quad &\Leftrightarrow \quad (x\ y\ z)A^{-1}A\left((x\ y\ z)A^{-1}\right)^\top \\ &\Leftrightarrow \quad \begin{pmatrix} x & y & z \end{pmatrix} A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \\ &\Leftrightarrow \quad [x,y,z] \in C_G \end{aligned}$$

with

$$G(X,Y,Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} A^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

This shows that $C^* = C_G$.

Exercise 13. Recall that $F(X,Y,Z) = 3Y^4 + 4Y^3Z + X^4$.

(a) We have $F(0,1,0) = 3 \neq 0$, i.e., $[0,1,0] \notin C_F$. This guarantees that the map $\pi : C_F \to \mathbb{P}^1_{\mathbb{C}}$, $\pi([a,b,c]) = [a,c]$ is well defined.

(b) We have $F_Y(X,Y,Z) = 12Y^2(Y+Z) = 0$. So the solutions of $F(P) = F_Y(P) = 0$ are given by

$$R = \{[0,0,1], [\pm 1, 1, -1], [\pm i, 1, -1]\} \subset C_F \cap C_{F_Y},$$

and

$$B = \pi(R) = \{[0,1], [\pm 1, -1], [\pm i, -1]\}.$$

We see that B contains 5 points. The y-coordinate of the each of the points in $\pi^{-1}([0,1])$ are given by the equation $y^3(3y+4) = 0$, so we have $|\pi^{-1}([0,1])| = 2$. The y-coordinate of each of the points in $\pi^{-1}([x,-1])$ with $x \in \{\pm 1, \pm i\}$ satisfies the equation $3y^4 + 4y^3 + 1 = (y+1)^2(3y^2 - 2y + 1) = 0$, so we have $|\pi^{-1}([x,-1])| = 3$ for all $x \in \{\pm 1, \pm i\}$.

(c) Since the singularities are a subset of R, we conclude from $F_X(X,Y,Z) = 3X^3$ and $F_Z(X,Y,Z) = 4Y^3$ that the only point in $\mathrm{Sing}(C_F)$ is $P = [0,0,1]$, which is one of the two points in $\pi^{-1}([0,1])$.

(d) We only need to carry out the blow-up procedure in the singular point P. We first choose affine coordinates via the identification $(x,y) \mapsto [x,y,1]$ and obtain the affine polynomial

$$f(x,y) = F(x,y,1) = 3y^4 + 4y^3 + x^4.$$

We see that we have a triple tangent line given by $y = 0$. So we can blow-up in U_0. We set $(x,y) = (x_1, x_1 y_1)$ and obtain

$$f(x_1, x_1 y_1) = x_1^3(3x_1 y_1^4 + 4y_1^3 + x_1),$$

so the strict transform of f in U_0 is

$$f^{(1)}(x_1, y_1) = 3x_1 y_1^4 + 4y_1^3 + x_1.$$

The preimages of $(x,y) = (0,0)$ under the strict transform are given by $x_1 = x = 0$ and $f^{(1)}(0, y_1) = 4y_1^3 = 0$, i.e., only $(x_1, y_1) = (0,0)$. This is a non-singular point of $C_{f^{(1)}}$ since

$$f^{(1)}_{x_1}(0,0) = 1.$$

So the blow-up process stops after one blow-up with a non-singular model $\psi : \widetilde{C} \to C_F$.

(e) Since B contains 5 points, we know from a result in the lectures that there exists a triangulation $\mathcal{T}$ of $\mathbb{P}^1_{\mathbb{C}}$ with the five points of B, and $3 \cdot 5 - 6 = 9$ edges and $2 \cdot 5 - 4 = 6$ triangles. The preimage $\pi^{-1}(B) \subset C_F$ contains $1 \cdot 2 + 4 \cdot 3 = 14$ points, and the preimage of P under the blow-up procedure $\psi : \widetilde{C} \to C_F$ consists of only one point. Since $\deg F = 4$, we end up with an induced triangulation of $\widetilde{C}$ with $V = 14$ vertices,

$E = 4 \cdot 9 = 36$ edges and $F = 4 \cdot 6 = 24$ triangles. This implies that $\widetilde{C}$ has the Euler number

$$\chi(\widetilde{C}) = V - E + F = 14 - 36 + 24 = 2.$$

(f) Using the relation $\chi(\widetilde{C}) = 2 - 2g(\widetilde{C})$, we conclude that the genus of the non-singular model $\widetilde{C}$ is

$$g(\widetilde{C}) = 1 - \frac{\chi(\widetilde{C})}{2} = 1 - \frac{2}{2} = 0.$$

Exercise 14. Recall that $F(X, Y, Z) = Y^4 - 2X^2Y^2 + XZ^3$.

(a) We have $F(0, 1, 0) = 1 \neq 0$, i.e., $[0, 1, 0] \notin C_F$. This guarantees that the map $\pi : C_F \to \mathbb{P}^1_{\mathbb{C}}$, $\pi([a, b, c]) = [a, c]$ is well defined.

(b) We have $F_Y(X, Y, Z) = 4Y(Y + X)(Y - X) = 0$. So the solutions of $F(P) = F_Y(P) = 0$ are given by

$$R = \{[0, 0, 1], [1, 0, 0], [\xi, \pm\xi, 1] \text{ with } \xi^3 = 1\} \subset C_F \cap C_{F_Y},$$

eight points in total, and

$$B = \pi(R) = \{[0, 1], [1, 0], [\xi, 1] \text{ with } \xi^3 = 1\},$$

five points in total. The y-coordinate of the each of the points in $\pi^{-1}([0, 1])$ are given by the equation $y^4 = 0$, so we have $|\pi^{-1}([0, 1])| = 1$. The y-coordinate of the each of the points in $\pi^{-1}([1, 0])$ are given by the equation $y^2(y^2 - 2) = 0$, so we have $|\pi^{-1}([0, 1])| = 3$. The y-coordinate of each of the points in $\pi^{-1}([\xi, 1])$ with $\xi^3 = 1$ satisfies the equation $y^4 - 2\xi^2y^2 + \xi = (y - \xi)^2(y + \xi)^2 = 0$, so we have $|\pi^{-1}([\xi, 1])| = 2$. So there are in total $1 + 3 + 3 \cdot 2 = 10$ points in $\pi^{-1}(B)$.

(c) Since the singularities are a subset of R, we conclude from $F_X(X, Y, Z) = -4XY^2 + Z^3$ and $F_Z(X, Y, Z) = 3XZ^2$ that the only point in $\mathrm{Sing}(C_F)$ is $P = [1, 0, 0]$, since $F_X(0, 0, 1) = 1 \neq 0$ and $F_Z(\xi, \pm\xi, 1) = 3\xi \neq 0$.

(d) We only need to carry out the blow-up procedure in the singular point P. We first choose affine coordinates via the identification $(x, y) \mapsto [1, x, y]$ and obtain the affine polynomial

$$f(x, y) = F(1, x, y) = x^4 - 2x^2 + y^3.$$

We see that we have a triple tangent line given by $x = 0$. So we need to blow-up in U_1. We set $(x, y) = (x_1 y_1, y_1)$ and obtain

$$f(x_1 y_1, y_1) = y_1^2(x_1^4 y_1^2 - 2x_1^2 + y_1),$$

so the strict transform of f in U_1 is

$$f^{(1)}(x_1, y_1) = x_1^4 y_1^2 - 2x_1^2 + y_1.$$

The preimages of $(x, y) = (0, 0)$ under the strict transform are given by $y_1 = y = 0$ and $f^{(1)}(x_1, 0) = -2x_1^2 = 0$, i.e., only $(x_1, y_1) = (0, 0)$. This is a non-singular point of $C_{f^{(1)}}$ since

$$f^{(1)}_{y_1}(0, 0) = 1.$$

So the blow-up process stops after one blow-up with a non-singular model $\psi : \widetilde{C} \to C_F$.

(e) Since B contains 5 points, we know from a result in the lectures that there exists a triangulation $\mathcal{T}$ of $\mathbb{P}^1_{\mathbb{C}}$ with the five points of B, and $3 \cdot 5 - 6 = 9$ edges and $2 \cdot 5 - 4 = 6$ triangles. The preimage $\pi^{-1}(B) \subset C_F$ contains 10 points, and the preimage of P under the blow-up procedure $\psi : \widetilde{C} \to C_F$ consists of only one point. Since $\deg F = 4$, we end up with an induced triangulation of $\widetilde{C}$ with $V = 10$ vertices, $E = 4 \cdot 9 = 36$ edges and $F = 4 \cdot 6 = 24$ triangles. This implies that $\widetilde{C}$ has the Euler number

$$\chi(\widetilde{C}) = V - E + F = 10 - 36 + 24 = -2.$$

(f) Using the relation $\chi(\widetilde{C}) = 2 - 2g(\widetilde{C})$, we conclude that the genus of the non-singular model $\widetilde{C}$ is

$$g(\widetilde{C}) = 1 - \frac{\chi(\widetilde{C})}{2} = 1 - \frac{-2}{2} = 2.$$

Exercise 15. Recall that $F(X,Y,Z) = X^5 + 3Y^5 - 5Y^3Z^2$.

(a) We have $F(0,1,0) = 3 \neq 0$, i.e., $[0,1,0] \notin C_F$. This guarantees that the map $\pi : C_F \to \mathbb{P}^1_{\mathbb{C}}$, $\pi([a,b,c]) = [a,c]$ is well defined.

(b) We have $F_Y(X,Y,Z) = 15Y^2(Y-Z)(Y+Z) = 0$. So the solutions of $F(P) = F_Y(P) = 0$ are given by

$$R = \{[0,0,1], [\alpha,1,1], [-\alpha,-1,1] \text{ with } \alpha^5 = 2\} \subset C_F \cap C_{F_y},$$

and

$$B = \pi(R) = \{[0,1], [\pm\alpha,1] \text{ with } \alpha^5 = 2\}.$$

We see that B contains 11 points and so does R. The y-coordinate of the each of the points in $\pi^{-1}([0,1])$ are given by the equation $y^3(3y^2 - 5) = 0$, so we have $|\pi^{-1}([0,1])| = 3$. The y-coordinate of each of the points in $\pi^{-1}([\alpha,1])$ with $\alpha^5 = 2$ satisfies the equation $3y^5 - 5y^3 + 2 = (y-1)^2(3y^3 + 6y^2 + 4y + 2) = 0$. Note that $y = 1$ is not a solution of $g(y) = 3y^3 + 6y^2 + 4y + 2$. Moreover, we have for the discriminant $D(g) = R(g,g')$, where $R(g,h)$ is the resultant of g,h,

$$D(g) = R(g,g') = \det \begin{pmatrix} 2 & 4 & 6 & 3 & 0 \\ 0 & 2 & 4 & 6 & 3 \\ 4 & 12 & 9 & 0 & 0 \\ 0 & 4 & 12 & 9 & 0 \\ 0 & 0 & 4 & 12 & 9 \end{pmatrix} = 900 \neq 0,$$

so $g(y)$ does not have multiple roots and we have $|\pi^{-1}([\alpha,1])| = 4$. A similar argument leads also to $|\pi^{-1}([-\alpha,1])| = 4$. So we have in total $1 \cdot 3 + 10 \cdot 4 = 43$ points in $\pi^{-1}(B) \subset C_F$.

(c) Since the singularities are a subset of R, we conclude from $F_X(X,Y,Z) = 5X^4$ and $F_Z(X,Y,Z) = -10Y^3Z$ that the only point in $\mathrm{Sing}(C_F)$ is $P = [0,0,1]$, which is one of the two points in $\pi^{-1}([0,1])$.

(d) We only need to carry out the blow-up procedure in the singular point P. We first choose affine coordinates via the identification $(x,y) \mapsto [x,y,1]$ and obtain the affine polynomial

$$f(x,y) = F(x,y,1) = x^5 + 3y^5 - 5y^3.$$

We see that we have a triple tangent line given by $y = 0$. So we can blow-up in U_0. We set $(x, y) = (x_1, x_1 y_1)$ and obtain

$$f(x_1, x_1 y_1) = x_1^3 (x_1^2 + 3x_1^2 y_1^5 - 5y_1^3),$$

so the strict transform of f in U_0 is

$$f^{(1)}(x_1, y_1) = x_1^2 + 3x_1^2 y_1^5 - 5y_1^3.$$

The preimages of $(x, y) = (0, 0)$ under the strict transform are given by $x_1 = x = 0$ and $f^{(1)}(0, y_1) = -5y_1^3 = 0$, i.e., only $(x_1, y_1) = (0, 0)$. This is still a singular point of $C_{f^{(1)}}$ since

$$f^{(1)}_{x_1}(x_1, y_1) = 2x_1 + 6x_1 y_1^5, \qquad f^{(1)}_{y_1}(x_1, y_1) = 15x_1^2 y_1^4 - 15y_1^2.$$

At $(0, 0) \in C_{f^{(1)}}$, we have again a double tangent line given by $x_1 = 0$. So we need to carry out the next blow-up again in U_1. We obtain

$$f^{(1)}(x_2 y_2, y_2) = y_2^2 (x_2^2 + 3x_2^2 y_2^5 - 5y_2),$$

so the strict transform of $f^{(1)}$ in U_1 is

$$f^{(2)}(x_2, y_2) = x_2^2 + 3x_2^2 y_2^5 - 5y_2.$$

The preimages of $(x_1, y_1) = (0, 0)$ under the strict transform are given by $y_2 = y_1 = 0$ and $f^{(2)}(x_2, 0) = x_2^2 = 0$, i.e., only $(x_2, y_2) = (0, 0)$. Since $f^{(2)}_{y_2}(x_2, y_2) = -5 \neq 0$, the point $(0, 0) \in C_{f^{(2)}}$ is no longer singular and the blow-up process stops with a non-singular model $\psi : \widetilde{C} \to C_F$.

(e) Since B contains 11 points, we know from a result in the lectures that there exists a triangulation $\mathcal{T}$ of $\mathbb{P}^1_{\mathbb{C}}$ with the 11 points of B, and $3 \cdot 11 - 6 = 27$ edges and $2 \cdot 11 - 4 = 18$ triangles. The preimage $\pi^{-1}(B) \subset C_F$ contains 43 points, and the preimage of P under the blow-up procedure $\psi : \widetilde{C} \to C_F$ consists of only one point. Since $\deg F = 5$, we end up with an induced triangulation of $\widetilde{C}$ with $V = 43$ vertices, $E = 5 \cdot 27 = 135$ edges and $F = 5 \cdot 18 = 90$ triangles. This implies that $\widetilde{C}$ has the Euler number

$$\chi(\widetilde{C}) = V - E + F = 43 - 135 + 90 = -2.$$

(f) Using the relation $\chi(\widetilde{C}) = 2 - 2g(\widetilde{C})$, we conclude that the genus of the non-singular model $\widetilde{C}$ is

$$g(\widetilde{C}) = 1 - \frac{\chi(\widetilde{C})}{2} = 1 - \frac{-2}{2} = 2.$$