Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 1 10.10.2011

The Web-Page of the course can be found at

http://www.maths.dur.ac.uk/∼dma0np/analysis1112/analysis.html

There you find the exercises, solutions, short summaries of forthcoming lectures, as well as other information.
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Do **Exercise 2** as **homework for this week**. The cumulative homework over the coming weeks will be collected and marked in a few weeks time.

You should definitively be able to do Exercise 3. Exercise 1 is a bit more demanding, but very rewarding.

1. **(p-adic numbers)** Let $p \in \mathbb{N}$ be a prime number. Any rational number $x = \frac{a}{b}$ can be written as $x = p^r \frac{a_0}{b_0}$, where a_0 and b_0 are not divisible by p. Define
$$\nu_p(p^r \frac{a_0}{b_0}) = p^{-r} \text{ and } \nu_p(0) = 0.$$
   (a) Show that $d_p(x, y) = \nu_p(x - y)$ defines a metric on $\mathbb{Q}$. For the triangle inequality, show first the following property of ν_p:
$$\nu_p(x + y) \leq \max\{\nu_p(x), \nu_p(y)\} \leq \nu_p(x) + \nu_p(y),$$
   and conclude that the following strong version of the triangle inequality holds, namely,
$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \leq d_p(x, y) + d_p(y, z).$$
   Remark: Note that two rationals are close in the metric d_p, if their difference is divisible by a high enough power of p. Moreover, the above inequality generalises to
$$\nu_p(\sum_{j=1}^{n} x_j) \leq \max\{\nu_p(x_j) \mid 1 \leq j \leq n\}.$$
   (b) Show that the sequence $x_n = p^n \in \mathbb{Q}$ is a null sequence, i.e., converges to zero.

(c) Let $2 \leq a \leq p-2$. Show that the sequence $x_n = a^{p^n}$ is a Cauchy sequence. **Hint:** Use Euler's Theorem

$$a^{\varphi(n)} \equiv 1 \mod n$$

for a, n co-prime, where φ is Euler's φ-function. Write x_n as the telescope sum $x_n = a + \sum_{j=0}^{n-1}(a^{p^{j+1}} - a^{p^j})$.

(d) Let x_n be the sequence in (c). Show that $x_n \not\to \pm 1$ and $x_n^{p-1} \to 1$. **Remark:** The Cauchy sequence x_n can be used to show that $(\mathbb{Q}, d_p)$ is not complete. If x_n were convergent in $\mathbb{Q}$, then it would converge to a $(p-1)$-th root of unity in $\mathbb{Q}$. But the only rational roots of unity are ± 1, which we ruled out as potential limits. The completion of $(\mathbb{Q}, d_p)$ yields the field $\mathbb{Q}_p$ of p-adic numbers. (Recall that the elements of $\mathbb{Q}_p$ are d_p-Cauchy sequences modulo zero sequences.)

2. Let $V = C[0,2] = \{f : [0,2] \to \mathbb{R} \,|\, f \text{ continuous}\}$ and we have an inner product on V given by

$$\langle f, g \rangle = \int_0^2 f(x)g(x)\,dx$$

for $f, g \in V$.

Define a sequence $(f_n)_{n\in\mathbb{N}}$ by

$$f_n(x) = \begin{cases} 0 & \text{for} \quad x \in [0, 1-1/n] \\ \frac{n}{2}x - \frac{n-1}{2} & \text{for} \quad x \in (1-1/n, 1+1/n) \\ 1 & \text{for} \quad x \in [1+1/n, 2] \end{cases}$$

(a) Show that $(f_n)_{n\in\mathbb{N}}$ is a sequence in V, i.e., show that each f_n is continuous.

(b) Show that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, where V has the metric induced from the inner product $\langle \cdot, \cdot \rangle$.

Remark: The Cauchy sequence f_n does not converge, so V is not complete. You should try to convince yourself that this is the case, but this is not part of the question.

3. (a) Let (X, d) be a metric space. Show that every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is a bounded set, i.e., there exists a constant $R > 0$ and a point $x \in X$ such that

$$d(x_n, x) \leq R \qquad \text{for all } n \in \mathbb{N},$$

i.e., all elements x_n lie in the closed ball $B_R(x)$ of radius R about x.

(b) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$. Show that the product $(x_n y_n)_{n\in\mathbb{N}}$ of the two Cauchy sequences is, again, a Cauchy sequence.