Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 10 14.12.2011

Do **Exercises 1, 2 and 4** as **homework** over Christmas. Exercise 4(b),(c) might be a bit more difficult, but is very worth trying. These homework exercises will not be marked, but you can check your solutions against the solution sheet. It is really important that you constantly work on homework questions to stay up to date with the course.

1. (Easy start) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x_1, x_2, x_3) = (y_1, y_2, y_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3^2).$$

Calculate the pullback $\omega = f^*(y_3 dy_1 \wedge dy_2 \wedge dy_3)$.

2. Let $U, V \subset \mathbb{R}^n$ be open and $\varphi : U \to V$ be a diffeomorphism with component functions $\varphi = (\varphi_1, \dots, \varphi_n)$. Let $x_1, \dots, x_n$ be the coordinates in U and $y_1, \dots, y_n$ in V. For $x \in U$, show that

$$(\varphi^*(dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n))_x = \det D\varphi(x)(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)_x.$$

Hint: You may use the fact that, for every permutation $\sigma \in \mathcal{S}_n$, we have

$$dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(n)} = \mathrm{sign}(\sigma) dx_1 \wedge \cdots \wedge dx_n,$$

and that the determinant of $A = (a_{ij})$ is given by $\det A = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$.

3. Let U_1, U_2, U_3 be three starlike subsets of $\mathbb{R}^n$. Suppose that the two intersections $U_1 \cap U_2$ and $U_2 \cap U_3$ are pathwise connected and $U_1 \cap U_3 = \emptyset$. Let $\omega \in \Omega^1(U_1 \cup U_2 \cup U_3)$ be closed. Show that ω is exact.

4. One important method in the calculation of de-Rham cohomologies is to start with the de-Rham cohomologies of easier domains and then to derive from them the de-Rham cohomologies of more complicated domains. A very useful tool in so doing is the *Mayer-Vietoris sequence*. Mayer-Vietoris allows us in many cases to calculate the deRham-cohomologies of a union of two sets from the knowledge of the de-Rham cohomologies of these two sets and their intersection. This exercise leads you through its core arguments. We may see later in the course (if time permits) how this can be used for calculations of de-Rham cohomologies. Let $V \subset U \subset \mathbb{R}^n$ be open. The restriction of a differential k-form $\omega \in \Omega^k(U)$ to V is denoted by $\omega|_V$.

Let $U_1, U_2 \subset \mathbb{R}^n$ be open and $U = U_1 \cup U_2$. Consider the sequence of maps

$$\Omega^k(U) \xrightarrow{I^k} \Omega^k(U_1) \times \Omega^k(U_2) \xrightarrow{J^k} \Omega^k(U_1 \cap U_2),$$

where $I^k(\omega) = (\omega|_{U_1}, \omega|_{U_2})$ and $J^k(\omega_1, \omega_2) = \omega_1|_{U_1 \cap U_2} - \omega_2|_{U_1 \cap U_2}$. Show the following facts:

(a) I^k is injective.

(b) We have

$$\operatorname{im} I^k = \{I^k(\omega) \mid \omega \in \Omega^k(U)\} = \ker J^k = \{(\omega_1, \omega_2) \mid J^k(\omega_1, \omega_2) = 0\}.$$

(c) J^k is surjective. For this part, you may use without proof that there exist two smooth functions $p_1, p_2 \in C^\infty(U)$ with $p_1, p_2 \geq 0$, $p_1 + p_2 = 1$, and

$$\operatorname{supp}(p_j) = \overline{\{x \in U \mid p(x) \neq 0\}} \subset U_j,$$

where $\overline{U} = U \cup \partial U$ denotes the closure of the set U. Such a family of functions p_1, p_2 is called a *partition of unity* for the open cover $\{U_1, U_2\}$ of U.

5. Let $U \subset \mathbb{R}^2$ be an open set and $F : U \to \mathbb{R}^2$ be a smooth vector field, i.e., $F(x, y) = (f(x, y), g(x, y))$ with $f, g \in C^\infty(U)$. Let $D \subset U$ be a closed disk of radius $r > 0$ around $p = (x_0, y_0) \in U$ and $c : [0, 2\pi] \to U$, $c(t) = p + (r\cos(t), r\sin(t))$ be a parametrisation of ∂D. Assume that F does not vanish at any point of ∂D. We call

$$n(F, D) := \frac{1}{2\pi} \int_c F^* \omega_0$$

the *index of F in D*, where $\omega_0 \in \Omega^1(U)$ was defined in Exercise 5 of Sheet 7. Geometrically, the index describes how many times the vector $F(x, y) \in \mathbb{R}^2 - 0$ rotates around the origin, as (x, y) runs once counter-clockwise around ∂D (and $n(F, D)$ is, therefore, closely related to the winding number and an integer). Prove the following fact:

If $n(F, D) \neq 0$, then there exists some point $q \in D$ such that $F(q) = 0$.

Hint: Assume that F doesn't have zeroes in D and introduce the free homotopy $H(t, s) = F((1 - s)c(t) + sp)$.

Remark: In fact, the invariant $n(F, D)$ **counts simple zeroes** of the vector field inside D. We call a point $p \in U$ with $F(p) = 0$ a *simple zero* of F, if $\det DF(p) \neq 0$. Since simple zeroes are isolated, there are only finitely many of them in compact disks. We call a simple zero p a *positive* zero if $\det DF(p) > 0$, and *negative* if $\det DF(p) < 0$. Assuming, $F : U \to \mathbb{R}^2$ has only simple zeroes in a disk $D \subset U$, none of which lying on ∂D. Then we have

$$n(F, D) = P - N,$$

where P is the number of positive zeroes in D and N the number of negative zeroes in D. (Of course, you don't need to prove this, even though the techniques are all there to do so!)

You may have seen an analogous concept in Complex Analysis, where a certain integral, namely $\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz$, counts the number of zeroes of a holomorphic function f. This analogy is another example for so many hidden non-trivial connections between the concepts of different courses, which makes maths so exciting and beautiful.

Merry Christmas and Happy New Year!!!

(taken from `http://www.netmums.com/activities/pictures-to-print/`)