Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 12 25.1.2012

The Problem Classes this term take place in CG85 at 5pm

on Thursday, 9 February 2012 and on Thursday, 8 March 2012.

Do **Exercises 2 and 4** as **homework for this week**. These homework exercises will not be marked, but you can check your solutions against the solution sheet in the following week. It is really important that you do every week the emphasized questions in order to stay up to date with the course.

1. Let $c : I = [a, b] \to \mathbb{R}^n$ be a smooth curve and $\omega \in \Omega^1(\mathbb{R}^m)$.

   (a) Let $m = n$. Then $c^*\omega \in \Omega^1(I)$. Show that

   $$\int_c \omega = \int_I c^*\omega,$$

   where the right hand side is understood as the integral of a 1-form over the one-dimensional rectangle I.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Then $f \circ c : I \to \mathbb{R}^m$. Show that

   $$\int_c f^*\omega = \int_{f\circ c} \omega,$$

   where both sides are integrals of 1-forms along curves.

2. Consider the ordinary differential equation

   $$x'(t) = F(x(t), t), \qquad x(0) = x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   with $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$,

   $$F(x, t) = \begin{pmatrix} tx_2 \\ -tx_1 \end{pmatrix}.$$

   Show that $F(x, t)$ is Lipschitz continuous in x in a neighbourhood of $(x_0, 0) \in \mathbb{R}^2 \times \mathbb{R}$. Calculate the first four Picard-Lindelöf iterations $x_1(t) = x_0$, $x_2(t)$, $x_3(t)$ and $x_4(t)$. Looking carefully at your results, can you guess the global solution of this initial value problem? Check your guess.

3. (a) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that

   $$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(b) Let A and B be $n \times n$ matrices with B invertible. Show that

$$e^{tB^{-1}AB} = B^{-1}e^{tA}B.$$

4. (a) Show that $f(x) = x^{1/2}$ does not satisfy a Lipschitz condition on any closed non-negative interval $I \subset \mathbb{R}$ containing 0.

   (b) Give an example of a continous function $g : \mathbb{R} \to \mathbb{R}$ which is not differentiable, but satisfies $|g(x) - g(y)| \leq L|x - y|$ for some $L \geq 0$ and all $x, y \in \mathbb{R}$.