Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 13 1.2.2012

Do **Exercise 2** as **homework for this week**. Exercise 4 is also interesting and highly recommended.

Your homework for this week and the cumulative homework over the previous weeks, i.e.,

- Exercise 2, Sheet 11
- Exercise 4, Sheet 11
- Exercise 2, Sheet 13

will be collected on Wednesday, 8 February, after the lecture. Do **not** submit any other homework questions, but check your solutions against the weekly distributed solution sheets.

1. The sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$ is a manifold of dimension n. Let $N = (0, \dots, 0, 1), S = (0, \dots, 0, -1) \in S^n$ For any two points $P, Q \in \mathbb{R}^{n+1}$, let

$$L_{P,Q} = \{\lambda P + (1 - \lambda)Q \mid \lambda \in \mathbb{R}\}$$

be the straight Euclidean line through P and Q. Two coordinate patches of S^n are given via stereographic projection by

$$\varphi_1 : \mathbb{R}^n \to S^n, \quad \varphi_1(x) = L_{(x,0),N} \cap (S^n - \{N\})$$

and

$$\varphi_2 : \mathbb{R}^n \to S^n, \quad \varphi_2(x) = L_{(x,0),S} \cap (S^n - \{S\}).$$

(You don't need to show this!) Calculate the images $\varphi_1(x_1, \dots, x_n)$ and $\varphi_2(x_1, \dots, x_n)$ explicitly as well as the coordinate change

$$\varphi_2^{-1} \circ \varphi_1 : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}.$$

Show that $D(\varphi_2^{-1} \circ \varphi_1)(x) = \frac{1}{\|x\|_2^2}\left(\mathrm{Id}_n - 2\frac{1}{\|x\|_2^2} x^\top x\right)$, where $x^\top x = (x_i x_j)_{1 \le i,j \le n}$. (The special case S^2 was discussed in the lecture.)

2. Let $F : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$F(x, y, z) = z^2 + \left(\sqrt{x^2 + y^2} - 5\right)^2.$$

(a) Show that 4 is a regular value of F, i.e., $M = F^{-1}(4)$ is a 2-dimensional manifold.

(b) Let $\varphi : U = (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ be defined by

$$\varphi(\alpha, \beta) = ((5 + 2\cos\beta)\cos\alpha, (5 + 2\cos\beta)\sin\alpha, 2\sin\beta).$$

Show that $\varphi(U) \subset M$, and check that φ is an almost global coordinate patch of M.

(c) Let $\omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy \in \Omega^2(\mathbb{R}^3)$. Calculate $\varphi^* dx$, $\varphi^* dy$, $\varphi^* dz$ and $\varphi^* \omega$.

3. Let $U \subset \mathbb{R}^k$ be open and $f : U \to \mathbb{R}$ be smooth. Show that the graph of f is a k-dimensional manifold in $\mathbb{R}^{k+1}$.

4. This exercise shows that the matrix group $SL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det A = 1\} \subset \mathbb{R}^{n^2}$ is a smooth manifold. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. From Exercise 3, Sheet 7, recall *Euler's relation*

$$\langle \operatorname{grad} f(x), x \rangle = m f(x),$$

where

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_k}(x) \right).$$

(a) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

(b) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that $SL(n, \mathbb{R})$ is a smooth manifold in $\mathbb{R}^{n^2}$. What is its dimension?