Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 15 22.2.2012

Do **Exercises 2 and 3** as **homework for this week**. These exercises are very useful to become acquainted with the *compatibitily condition* and with the concept of *orientability.*

This homework will be collected on Wednesday, 29 February, after the lecture. Do **not** submit any other homework questions, but check your solutions with the solution sheets.

There is also still strong need for most of you to exercise on *regular values.* This is the topic of the highly recommended Exercise 1.

1. Let $A := \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0, z \neq 0\}$ and $f : A \to \mathbb{R}^3$ be defined by
$$f(x,y,z) = (x^2 - y^3, yz, z^3).$$
   (a) Show that
$$\mathrm{im}(f) = f(A) = \left\{ (u,v,w) \in \mathbb{R}^3 \mid w \neq 0, u \geq -\frac{v^3}{w} \right\}.$$
   (b) Show that $(u,v,w) \in \mathbb{R}^3$ is a regular value of f if and only if
$$(w = 0) \quad \text{or} \quad \left(w \neq 0 \text{ and } u \neq -\frac{v^3}{w} \right).$$

2. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid \|(x,y,z)\|_2 = 1\}$, and $\varphi_\alpha, \varphi_\beta : \mathbb{R}^2 \to S^2$ be an atlas of stereographic projections, i.e.,
$$\begin{aligned} \varphi_\alpha(u_1,u_2) &= \frac{1}{u_1^2+u_2^2+1}(2u_1, 2u_2, u_1^2+u_2^2-1), \\ \varphi_\beta(v_1,v_2) &= \frac{1}{v_1^2+v_2^2+1}(2v_1, 2v_2, 1-v_1^2-v_2^2). \end{aligned}$$
   Recall that the coordinate change $\varphi_\beta^{-1} \circ \varphi_\alpha : \mathbb{R}^2 - 0 \to \mathbb{R}^2 - 0$ is given by
$$\varphi_\beta^{-1} \circ \varphi_\alpha(u_1,u_2) = \frac{1}{u_1^2+u_2^2}(u_1,u_2).$$
   (a) Let
$$\begin{aligned} \omega_\alpha &= \frac{1}{u_1^2}(u_1 du_1 + u_2 du_2), \\ \omega_\beta &= -\frac{1}{v_1^2}(v_1 dv_1 + v_2 dv_2). \end{aligned}$$
   Then $\omega_\alpha, \omega_\beta \in \Omega^1(\mathbb{R}^2 - 0)$. Check that these two differential forms satisfy the compatibility condition, i.e., are the pullbacks of a globally defined differential 1-form on S^2.

   (b) Check that the differential forms $d\omega_\alpha$ and $d\omega_\beta$ from (a) are the pullbacks of the global differential form $-\frac{2}{x^3} dx \wedge dz$ on S^2.

3. (a) Let $v = (x, y, z)^\top$ and $w = (a, b, c)^\top$ be linear independent. Show that the ordered basis $v \times w, v, w$ carries the same orientation as e_1, e_2, e_3.

(b) Let

$$\varphi : (-1, 1) \times \mathbb{R} \to \mathbb{R}^3, \quad \varphi(s, t) = ((2 + s) \cos t, (2 + s) \sin t, s^2)$$

and $M = \varphi((-1, 1) \times [0, 2\pi])$. We assume M carries the orientation given by the atlas consisting of the two local coordinate patches

$$\begin{aligned} \varphi_1 &= \varphi\Big|_{(-1,1)\times(0,2\pi)} : (-1, 1) \times (0, 2\pi) \to \mathbb{R}^3, \\ \varphi_2 &= \varphi\Big|_{(-1,1)\times(-\pi,\pi)} : (-1, 1) \times (-\pi, \pi) \to \mathbb{R}^3. \end{aligned}$$

You don't need to prove that the coordinate change is orientation preserving. Find an implicit description of M, i.e., find a function

$$f : \{(x, y, z) \mid 1 < \sqrt{x^2 + y^2} < 3\} \to \mathbb{R}$$

such that 0 is a regular value of f and $M \subset f^{-1}(0)$. You don't need to prove in full that $M = f^{-1}(0)$.

(c) Let M be the manifold in (b). Show that e_3 is a unit normal vector of M at the point $(2, 0, 0) \in M$. Decide whether e_3 is positively oriented with respect to the orientation induced by the atlas $\{\varphi_1, \varphi_2\}$.