Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 2 17.10.2011

The Problem Classes this term take place in CG85 at 5pm

on Thursday, 20 October 2011 and on Thursday, 17 November 2011.

Do **Exercises 1 and 2** as **homework for this week**. These exercise will not be marked, but you can check your solutions against the solution sheet in the following week. It is really important that you do every week the emphasized questions in order to stay up to date with the course.

You should definitively be able to do Exercise 3. Exercise 4 requires an Induction proof. Exercise 5 introduces the important space $l^1(\mathbb{C})$ of absolutely summable sequences.

1. **"Nested Subsets Principle"** Let (M, d) be a complete metric space. For a subset $A \subset M$, let

   $$\mathrm{diam}(A) = \sup\{d(p,q) \mid p, q \in A\}$$

   be the *diameter of* A.

   (a) Let K_j be a sequence of non-empty closed subsets of M with the following properties:

      (i) The K_j are nested, i.e., we have $K_{j+1} \subset K_j$.

      (ii) We have $\mathrm{diam}(K_j) \to 0$ as $j \to \infty$.

      Prove the following fact: The intersection $\bigcap_{j=1}^{\infty} K_j$ is non-empty and consists of a single point in M.

      Hint: Choose an arbitrary sequence (x_j) with $x_j \in K_j$. Show that (x_j) is a Cauchy sequence. This brings you to the right track to successfully find your proof of this principle.

   (b) Find an example of nested open sets U_j in a complete metric space with the properties (i) and (ii) above for which we have $\bigcap_j U_j = \emptyset$. So the closedness-condition of the sets is essential for the Nested Subsets Principle.

2. For $f, g \in B([a,b])$ let $d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$ and $C([a,b]) \subset B([a,b])$ be the subspace of continuous functions. Assume $f_n \in C([a,b])$ is a sequence with $f_n \to f \in B([a,b])$. Prove that the limit f is continuous.

   In other words: *A uniformly convergent sequence of continuous functions has a continuous limit.*

   Hint: Use the (ϵ, δ)-definition of continuity and the fact that f_n is uniformly close to f, for large enough n, and that f_n is continuous. Combine all this to create a precise proof.

3. Let (M,d) be a metric space and $(A_\alpha)_{\alpha\in I}$ be a family of open subsets of M. Show that $\bigcup_{\alpha\in I} A_\alpha$ is open.

4. Let $f_j \in C([0,1])$ be defined by $f_j(t) = t^j$ and

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Show that the sequence $h_j \in C([0,1])$, defined by

$$g_j = f_j - \sum_{k=0}^{j-1} \langle f_j, h_k\rangle h_k, \qquad h_j = \frac{1}{\|g_j\|} g_j$$

with $\|f\| = (\langle f, f\rangle)^{1/2}$, is a family of pairwise orthogonal functions of norm one (i.e., $\langle f_j, f_k\rangle = \delta_{jk}$). This procedure is called *Gram-Schmidt Orthogonalisation.* Calculate h_0, h_1, h_2.

5. Let $l_1(\mathbb{C}) = \{\mathbf{s} = (x_n) \subset \mathbb{C} \mid \sum_{n=1}^\infty |x_n| < \infty\}$. $l_1(\mathbb{C})$ is a vector space with metric

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^\infty |x_n - y_n|,$$

for $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$. Let $\mathbf{0} = (0, 0, 0, \dots) \in l_1(\mathbb{C})$ be the zero sequence. Show that the closed unit ball

$$B_1(\mathbf{0}) = \{\mathbf{s} \in l_1(\mathbb{C}) \mid d(\mathbf{s}, \mathbf{0}) \le 1\}$$

is not compact. This means that the Theorem of Heine-Borel (Theorem 1.21) does not hold for this space.

Hint: Look at the sequences $\mathbf{s}_1 = (1, 0, 0, 0, \dots)$, $\mathbf{s}_2 = (0, 1, 0, 0, \dots)$, $\mathbf{s}_3 = (0, 0, 1, 0, \dots)$ and so on in the unit ball $B_1(\mathbf{0})$.