Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 4 31.10.2011

Do **Exercises 1 and 4** as **homework for this week**. Exercise 3 is a good exercise to train your abilities to carry out proofs. Exercise 5 is useful in connection with the Contraction Mapping Principle. These homework exercises will not be marked, but you can check your solutions against the solution sheet in the following week. It is really important that you do every week the emphasized questions in order to stay up to date with the course.

1. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V and $\alpha, \beta > 0$. Show that the linear combination
$$\|x\| := \alpha\|x\|_1 + \beta\|x\|_2$$
is again a norm. Determine the shape of the closed unit ball $B_1(0)$ of the norm
$$\|x\| = \frac{1}{3}\left(|x_1| + |x_2|\right) + \frac{2}{3}\max\{|x_1|, |x_2|\}.$$

2. Let p be a prime number and ν_p the function introduced in Exercise 1 on Sheet 1. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the norm ν_p (analogously as $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the absolute value norm $|\cdot|$). Let $\sum_{k=0}^{\infty} a_k$ be a formal sequence of rational numbers $a_k \in \mathbb{Q}$. Show the following fact: $\sum_{k=1}^{\infty} a_k$ convergent in $\mathbb{Q}_p$ if and only if $a_k \to 0$ with respect to the ν_p norm. Note that an analogous fact does not hold in $\mathbb{R}$: there $\sum_{k=1}^{\infty} a_k$ convergent implies $a_k \to 0$, but not conversely (think of the sequence $\sum_{k=1}^{\infty} \frac{1}{k}$).

   Hint: In the completion $\mathbb{Q}_p$ convergence of $\sum a_k$ is equivalent to the sequence $A_n := \sum_{k=1}^{n} a_k$ being a Cauchy sequence. Use the strong triangle inequality.

3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces and $\mathcal{B}(V, W)$ be the vector space of bounded linear operators with the operator norm
$$\|T\| := \sup\{\|Tx\|_W \mid x \in V \text{ with } \|x\|_V \leq 1\}.$$
Show that $(\mathcal{B}(V, W), \|\cdot\|)$ is a Banach space if $(W, \|\cdot\|_W)$ is a Banach space.

4. Define $C^1[a, b]$ to be the subspace of $C[a, b]$ consisting of those $f : [a, b] \to \mathbb{R}$, which have a continuous derivative $f' : [a, b] \to \mathbb{R}$ (here $f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ is assumed to exist, and similarly with $f'(b)$. For $f \in C[a, b]$, let $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.

   (i) Show that the linear operator $D : C^1[a, b] \to C[a, b]$, $D(f) = f'$ is not bounded if both vector spaces $C^1[a, b]$ and $C[a, b]$ carry the supremum norm $\|\cdot\|_\infty$.

(ii) Let $P_k[a,b]$ be the vector space of real polynomials of degree $\leq k$ on $[a,b]$. Show that the above operator $D : P_k[a,b] \to P_k[a,b]$ (restricted to $P_k[a,b]$) is bounded if $P_k[a,b]$ carries the supremum norm $\|\cdot\|_\infty$.

(iii) Show that $\|\cdot\|_* : C^1[a,b] \to [0,\infty)$, defined by

$$\|f\|_* = \|f'\|_\infty,$$

does not give a norm on $C^1[a,b]$.

(iv) For $f \in C[0,1]$, define

$$\|f\|_\triangle = \sum_{j=0}^{k} |f(\frac{j}{k})|.$$

Is $\|\cdot\|_\triangle$ a norm on $C[0,1]$? Is it a norm on $P_k[0,1]$?

(v) Show that $\|\cdot\|_{C^1} : C^1[a,b] \to [0,\infty)$, defined by

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$$

gives a norm on $C^1[a,b]$ and that the above operator D is bounded if $C^1[a,b]$ and $C[a,b]$ carry the norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_\infty$, respectively.

(vi) Is $\|\cdot\|_\diamond : C^1[a,b] \to [0,\infty)$, defined by

$$\|f\|_\diamond = \|f'\|_\infty + |f(a)|$$

a norm on $C^1[a,b]$.

5. Let $f : [a,b] \to [a,b]$ be a continuous function.

(i) Use the Intermediate Value Theorem to show that f has a fixed point, i.e., a point $x \in [a,b]$ with $f(x) = x$.

(ii) Assume that $f : [a,b] \to [a,b]$ is C^1 and that $|f'(x)| < 1$ for all $x \in [a,b]$. Show that, for every $x_0 \in [a,b]$, the sequence x_n with $x_{n+1} = f(x_n)$ (for all n) is converging to a fixed point of f.
Hint: Use the Contraction Mapping Principle.

(iii) Find a function $f : [a,b] \to [a,b]$ which is C^1 and $\|f'\|_\infty \leq 1$, such that there is a sequence x_n with $x_{n+1} = f(x_n)$ (for all n) which is non convergent.