Analysis III/IV (Math 3011, Math 4201)

Exercise Sheet 5 7.11.2011

Do **Exercise 3** as **homework for this week**. Exercises 1 and 2 should be relatively easy, but instructive.

Your homework for this week and the cumulative homework over the previous weeks, i.e.,

- Exercise 2, Sheet 1
- Exercise 2, Sheet 3
- Exercise 3, Sheet 5

will be collected on Monday, 14 November, after the lecture. Do **not** submit any other homework questions, but check your solutions against the weekly distributed solution sheets.

1. (Easy Warmup) Let (M, d) be a complete metric space and $x_n \in M$ a sequence satisfying $d(x_n, x_{n+1}) \leq \frac{1}{n^2}$. Show that x_n is convergent.

2. Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq 0, \\ 0 & \text{if } (x,y) = 0. \end{cases}$$

Show that f is discontinuous at $(x, y) = 0$ (consider the behaviour of the function along different straight lines through the origin) but, at the same time, that f has globally well defined first partial derivatives. Can these partial derivatives be continuous?

3. Let $I \subset \mathbb{R}$ be an open interval and $F : \mathbb{R} \times I \to \mathbb{R}$ be continuous and Lipschitz continuous in the first variable, i.e., there exists $L > 0$ such that

$$|F(x_1, t) - F(x_2, t)| \leq L|x_1 - x_2| \qquad \text{for all } x_1, x_2 \in \mathbb{R} \text{ and all } t \in I.$$

Let $t_0 \in I$ and $x_0 \in \mathbb{R}$ and, for $\epsilon > 0$, let $I_\epsilon := [t_0 - \epsilon, t_0 + \epsilon]$. Show that, for $\epsilon > 0$ small enough, the map

$$T : C(I_\epsilon) \to C(I_\epsilon),$$

defined by

$$Tf(t) := x_0 + \int_{t_0}^{t} F(f(s), s)\, ds$$

is a contraction in the complete metric space $C(I_\epsilon)$ with metric $d_\infty(f, g) = \|f - g\|_\infty$.

4. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces and $T : V \to W$ be a linear map. Prove that the following properties are equivalent:

   (i) T is a bounded linear operator.

   (ii) T is continuous at $v = 0 \in V$.

   (iii) T is a continuous map everywhere.

5. For $p \geq 1$ let $l_p(\mathbb{C}) = \{\mathbf{x} = (x_n) \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Note that this is a vector space since for and $x, y \geq 0$, we have

$$(x+y)^p \leq (x+x)^p + (y+y)^p = 2^n(x^n + y^n).$$

We define a norm $\|\cdot\|_p$ on $l_p(\mathbb{C})$ as follows:

$$\|\mathbf{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

Our aim is to prove that this norm satisfies the triangle inequality for $p > 1$. You may use without proof that, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for $x, y \geq 0$, we have

$$x^{1/p} y^{1/q} \leq \frac{x}{p} + \frac{y}{q}. \tag{1}$$

(This follows from $\ln''(x) = -1/x^2 < 0$ and is an application of the concavity of the logarithm function.)

   (i) Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{x} = (x_n) \in l_p(\mathbb{C})$ and $\mathbf{y} = (y_n) \in l_q(\mathbb{C})$. Show **Hölder's Inequality**, i.e.,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

   Hint: Define $\xi_n = \frac{|x_n|^p}{\|\mathbf{x}\|_p^p}$ and $\eta_n = \frac{|y_n|^q}{\|\mathbf{y}\|_q^q}$ and apply (1) to ξ_n and η_n.

   (ii) Let $p > 1$ and $\mathbf{x} = (x_n), \mathbf{y} = (y_n) \in l_p(\mathbb{C})$. Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{z} = (z_n)$ with $z_n = |x_n + y_n|^{p-1}$. Show that $\mathbf{z} \in l_q(\mathbf{C})$.

   (iii) Derive

$$\|\mathbf{x} + \mathbf{y}\|_p^p \leq \sum |x_n| \cdot |z_n| + \sum |y_n| \cdot |z_n|,$$

   and apply Hölder's inequality to the terms on the right hand side to obtain **Minkowski's Inequality**, namely,

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

   Remark: Note that Minkowski's Inequality is just the triangle inequality in the normed vector space $l_p(\mathbf{C})$. Moreover, Hölder's Inequality in the special case $p = q = 2$ is just Cauchy-Schwarz. We already saw in the lectures that Cauchy-Schwarz implies the triangle inequality.