**Analysis III/IV (Math 3011, Math 4201)**

**Solutions to Exercise Sheet 10** 14.12.2011

1. Let $f = (f_1, f_2, f_3)$. We have

$$f^*(y_3 dy_1 \wedge dy_2 \wedge dy_3) = x_3^2 (df_1)_x \wedge (df_2)_x \wedge (df_3)_x =$$
$$x_3^2(\cos x_2 dx_1 - x_1 \sin x_2 dx_2) \wedge (\sin x_2 dx_1 + x_1 \cos x_2 dx_2) \wedge 2x_3 dx_3 =$$
$$2x_3^3(x_1 \cos^2 x_2 + x_1 \sin^2 x_2) dx_1 \wedge dx_2 \wedge dx_3 = 2x_1 x_3^3 dx_1 \wedge dx_2 \wedge dx_3.$$

2. We have

$$\varphi^*(dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n) = d\varphi_1 \wedge d\varphi_2 \wedge \cdots \wedge d\varphi_n =$$
$$\left( \sum_{i_1=1}^n \frac{\partial \varphi_1}{\partial x_{i_1}} dx_{i_1} \right) \wedge \cdots \wedge \left( \sum_{i_n=1}^n \frac{\partial \varphi_n}{\partial x_{i_n}} dx_{i_n} \right) =$$
$$\sum_{\sigma \in \mathcal{S}_n} \frac{\partial \varphi_1}{\partial x_{\sigma(1)}} \frac{\partial \varphi_2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi_n}{\partial x_{\sigma(n)}} dx_{\sigma(1)} \wedge dx_{\sigma(2)} \wedge \cdots \wedge dx_{\sigma(n)} =$$
$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma) \frac{\partial \varphi_1}{\partial x_{\sigma(1)}} \frac{\partial \varphi_2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi_n}{\partial x_{\sigma(n)}} dx_1 \wedge \cdots \wedge dx_n = \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{pmatrix} dx_1 \wedge \cdots \wedge dx_n.$$

Evaluation at the point $x \in U$ yields

$$(\varphi^*(dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n))_x = \det D\varphi(x)(dx_1 \wedge \cdots \wedge dx_n)_x.$$

3. The restriction $\omega|_{U_j}$ of $\omega$ to any of the domains $U_j$ is exact, so we have $\omega|_{U_j} = df_j$ with $f_j \in C^\infty(U_j)$. On $U_{12} := U_1 \cap U_2$, we have $df_1|_{U_{12}} = df_2|_{U_{12}}$, so $(df_1 - df_2)|_{U_{12}} = 0$. Since $U_{12}$ is pathwise connected, and the differential of $(f_1 - f_2)|_{U_{12}}$ vanishes, the function $(f_1 - f_2)|_{U_{12}}$ must be constant. Let $(f_1 - f_2)|_{U_{12}} = c_{12} \in \mathbb{R}$. Similar arguments show that $(f_2 - f_3)|_{U_{23}} = c_{23} \in \mathbb{R}$. Define $f \in C^\infty(U_1 \cup U_2 \cup U_3)$ as follows:

$$f(x) = \begin{cases} f_1(x) - c_{12} & \text{if } x \in U_1 \\ f_2(x) & \text{if } x \in U_2 \\ f_3(x) + c_{23} & \text{if } x \in U_3 \end{cases}$$

The function is obviously well defined on $U_1 \cup U_2 \cup u_3$. Moreover, we have

$$df_x = \begin{cases} (df_1)_x & \text{if } x \in U_1 \\ (df_2)_x & \text{if } x \in U_2 \\ (df_3)_x & \text{if } x \in U_3 \end{cases} = \omega_x,$$

i.e., $\omega \in \Omega^1(U_1 \cup U_2 \cup U_3)$ is exact.

4. (a) Assume that $I^k(\omega) = 0$. Then $\omega_{U_1} = 0$ and $\omega_{U_2} = 0$. Since any point $x \in U$ lies in either $U_1$ or $U_2$, this implies $\omega_x = 0$, i.e., $\omega = 0$ and $I^k$ is injective.

(b) Let $\omega \in \Omega^k(U)$. Then

$$J^k(I^k(\omega)) = J^k(\omega|_{U_1}, \omega|_{U_2}) = (\omega|_{U_1})|_{U_1\cap U_2} - (\omega|_{U_2})|_{U_1\cap U_2} = \omega|_{U_1\cap U_2} - \omega|_{U_1\cap U_2} = 0.$$

This shows that $\operatorname{im} I^K \subset \ker J^k$. Now, we assume that $(\omega_1, \omega_2) \in \ker J^k$. Let $\omega_1 = \sum_I f_I dx_I$, $\omega_2 = \sum_I g_I dx_I$, where $I = (i_1, \dots, i_k)$ runs through all multi-indices with $i_1 < i_2 < \cdots < i_k$. We conclude from $J^k(\omega_1, \omega_2) = 0$ that $\omega_1|_{U_1\cap U_2} = \omega_2|_{U_1\cap U_2}$, which implies that $f_I|_{U_1\cap U_2} = g_I|_{U_1\cap U_2}$. Now, define functions $h_I \in C^\infty(U)$ by

$$h_I(x) = \begin{cases} f_I(x), & \text{if } x \in U_1 \\ g_I(x), & \text{if } x \in U_2 \end{cases}$$

$h_I$ is well-defined since the restrictions of $f_I$ and $g_I$ on the intersection $U_1 \cap U_2$ agree. Then $\omega = \sum_I h_I dx_I \in \Omega^k(U)$ is well-defined, and we obviously have

$$I^k(\omega) = (\omega|_{U_1}, \omega|_{U_2}) = (\sum_I f_I dx_I, \sum_I g_I dx_I) = (\omega_1, \omega_2).$$

This shows the converse inclusion $\ker J^k \subset \operatorname{im} I^k$.

(c) For a given function $f \in C^\infty(U_1 \cap U_2)$, let us introduce smooth extensions of $f$ to $C^\infty(U_1)$ and $C^\infty(U_2)$ via

$$f^1(x) = \begin{cases} p_2(x)f(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_1 - U_2 \end{cases} \qquad f^2(x) = \begin{cases} p_1(x)f(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_2 - U_1 \end{cases}$$

Then we obviously have $(f^1 + f^2)|_{U_1\cap U_2} = p_1|_{U_1\cap U_2} f + p_2|_{U_1\cap U_2} f = (p_1 + p_2)|_{U_1\cap U_2} f = f$. For a differential form $\omega \in \Omega^k(U_1 \cap U_2)$, given by

$$\omega = \sum_I f_I dx_I,$$

we define $\omega_1 = \sum_I {f_I}^1 dx_I \in \Omega^k(U_1)$ and $\omega_2 = -\sum_I {f_I}^2 dx_I \in \Omega^k(U_2)$. Then

$$J^k(\omega_1, \omega_2) = \sum_I \left(f_I^1 - (-f_I^2)\right)|_{U_1\cap U_2} dx_I = \sum_I f_I dx_I = \omega,$$

which shows that $J^k$ is surjective.

5. Assume that $F$ doesn't have zeroes in $D$. Since $c : [0, 2\pi] \to U$ is a closed curve, so is $\gamma = F \circ c : [0, 2\pi] \to \mathbb{R}^2 - 0$. Let $\gamma_0 : [0, 2\pi] \to \mathbb{R}^2 - 0$ be the point curve $\gamma_0(t) = F(p)$. Observe that

$$(1 - s)c(t) + sp = p + (1 - s)r(\cos t, \sin t) \in D$$

for all $s \in [0,1]$. This shows that the map $H(t,s) = F((1-s)c(t)+sp)$ is well defined. Moreover, it is a free homotopy between $\gamma$ and $\gamma_0$, since $H(t,0) = F(c(t)) = \gamma(t)$, $H(t,1) = F(p) = \gamma_0(t)$ and

$$H(0,s) = F((1-s)c(0)+sp) = F((1-s)c(2\pi)+sp) = H(2\pi,s).$$

Next, we calculate

$$\int_c F^*\omega_0 = \int_0^{2\pi} (F^*\omega_0)_{c(t)}(c'(t))dt = \int_0^{2\pi} (\omega_0)_{F(c(t))}(DF(c(t))c'(t))dt = \int_0^{2\pi} (\omega_0)_{F\circ c(t)}(F\circ c)'(t)dt = \int_0^{2\pi} (\omega_0)_{\gamma(t)}(\gamma'(t))dt = \int_\gamma \omega_0.$$

We know from Exercise 2(b) on Sheet 7 that $\omega_0$ is closed. Since $\gamma$ and $\gamma_0$ are freely homotopic, we conclude that

$$\int_\gamma \omega_0 = \int_{\gamma_0} \omega_0 = \int_0^{2\pi} (\omega_0)_{F(p)} \underbrace{(\gamma_0'(t))}_{=0} dt = 0.$$

This obviously contradicts to $n(F,D) \neq 0$.