Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 12 25.1.2012

1. (a) Let $\omega = \sum_{i=1}^n f_i dx_i$. Then we have $c^* dx_i = c_i'(t)dt$ and $c^*\omega = \sum_{i=1}^n (f_i \circ c)c_i' dt$. This implies that

$$\int_c \omega = \int_a^b \sum_i (f_i \circ c)c_i'(t)dt = \int_{[a,b]} \sum_i (f_i \circ c)c_i'(t)dt = \int_I c^*\omega,$$

where the last equation is just the definition of the integration of a one-form over the 1-dimensional rectangle $I = [a,b] \subset \mathbb{R}$.

(b) We have

$$\begin{aligned}\int_c f^*\omega = \int_a^b (f^*\omega)_{c(t)}(c'(t))dt &= \int_a^b \omega_{f\circ c(t)}(Df(c(t))c'(t))dt \\ &= \int_a^b \omega_{f\circ c9t)}(f\circ c)'(t)dt = \int_{f\circ c} \omega.\end{aligned}$$

2. We first check Lipschitz continuity:

$$\begin{aligned}\|F(x,t) - F(y,t)\|_2 = \| \begin{pmatrix} t(x_2 - y_2) \\ -t(x_1 - y_1) \end{pmatrix} \|_2 = \\ t\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = t\|x-y\|_2.\end{aligned}$$

This implies that $F : \mathbb{R}^2 \times (-T,T) \to \mathbb{R}^2$ is Lipschitz continuous with $L = T$. The Picard-Lindelöf iterations are

$$\begin{aligned}
x_1(t) &= x_0, \\
x_2(t) &= x_0 + \int_0^t \begin{pmatrix} s \\ 0 \end{pmatrix} ds = \begin{pmatrix} t^2/2 \\ 1 \end{pmatrix}, \\
x_3(t) &= x_0 + \int_0^t \begin{pmatrix} s \\ -s^3/2 \end{pmatrix} ds = \begin{pmatrix} t^2/2 \\ 1 - t^4/8 \end{pmatrix}, \\
x_4(t) &= x_0 + \int_0^t \begin{pmatrix} s - s^5/8 \\ -s^3/2 \end{pmatrix} ds = \begin{pmatrix} t^2/2 - t^6/(2^3 \cdot 3!) \\ 1 - t^4/(2^2 \cdot 2!) \end{pmatrix}.
\end{aligned}$$

An educated guess of the solution would be

$$x(t) = \begin{pmatrix} \sin(t^2/2) \\ \cos(t^2/2) \end{pmatrix},$$

and one easily checks

$$x'(t) = \begin{pmatrix} t\cos(t^2/2) \\ -t\sin(t^2/2) \end{pmatrix} = F(x(t),t)$$

and

$$x(0) = x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

3. (a) We have $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and so $A^{2k} = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix}$ and $A^{2k+1} = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix}$.
Therefore we get

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \end{pmatrix} + \begin{pmatrix} 0 & -\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & -\sin t \\ \sin t & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

(b) Note that

$$(B^{-1}AB)^k = (B^{-1}AB)(B^{-1}AB)\cdots(B^{-1}AB) = B^{-1}A^kB$$

and therefore

$$\begin{aligned} e^{tB^{-1}AB} &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(B^{-1}AB)^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} B^{-1}A^kB \\ &= B^{-1}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right)B \\ &= B^{-1}e^{tA}B. \end{aligned}$$

4. (a) Assume $|f(x) - f(y)| \leq L \cdot |x - y|$ for all $x, y \in I$ with $I \subset \mathbb{R}$ non-negative and closed and $0 \in I$ and $f(0) = 0$. Then in particular $f(x) \leq L \cdot x$ for $x \in I$. So if $x \in I$, we get

$$\begin{aligned} \sqrt{x} &\leq L \cdot x \\ \frac{1}{\sqrt{x}} &\leq L \\ \frac{1}{L^2} &\leq x, \end{aligned}$$

but the last line can be violated by choosing $0 < x < \frac{1}{L^2}$. Therefore f does not satisfy a Lipschitz condition near 0.

(b) Choose $g(x) = |x|$, then g is not differentiable at $x = 0$, but

$$|g(x) - g(y)| = |\, |x| - |y|\, | \leq |x - y|$$

so g satisfies a Lipschitz condition with $L = 1$.