Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 15 22.2.2012

1. (a) Let $(u, v, w) = f(x, y, z)$. Since $z \neq 0$, we have $w \neq 0$ and

$$u = x^2 - y^3 \geq -y^3 = -\frac{(yz)^3}{z^3} = -\frac{v^3}{w}.$$

Conversely, let (u, v, w) satisfy $w \neq 0$ and $u \geq -v^3/w$. Since $w \neq 0$, we can choose $z = w^{1/3} \neq 0$ (z has the same sign as w, so $z = w^{1/3}$ if $w > 0$ and $z = -|w|^{1/3}$ if $w < 0$), and $y = v/z = v/(w^{1/3})$ and $x = \sqrt{u + y^3} = \sqrt{u + v^3/w}$ (the square root is well defined since $u \geq -v^3/w$), and we obviously have $f(x, y, z) = (u, v, w)$.

(b) We have

$$Df(x, y, z) = \begin{pmatrix} 2x & -3y^2 & 0 \\ 0 & z & y \\ 0 & 0 & 3z^2 \end{pmatrix}$$

and $\det Df(x, y, z) = 6xz^3$. This means that (x, y, z) is a singular point of f if and only if $xz = 0$. Since for points (x, y, z) in the domain of f we have $z \neq 0$, the singular points of f are precisely the points $(0, y, z) \in A$. Now,

$$f(0, y, z) = (-y^3, yz, z^3),$$

i.e., $(u, v, w) \in \mathbb{R}$ is a regular value of f if and only if $(u, v, w) \neq (-y^3, yz, z^3)$ for any choice of $y \in \mathbb{R}$ and $z \neq 0$. Since $z \neq 0$, any value (u, v, w) with $w = 0$ is automatically regular.

Assume $w \neq 0$. If $(u, v, w) = (-y^3, yz, z^3)$ for some $y \in \mathbb{R}$ and $z \neq 0$ then, necessarily,

$$-\frac{v^3}{w} = -\frac{(yz)^3}{z^3} = -y^3 = u.$$

Conversely, if $u = -v^3/w$ then, choosing $z = w^{1/3} \neq 0$, $y = v/z = v/(w^{1/3})$, we have

$$(-y^3, yz, z^3) = \left(-\frac{v^3}{w}, v, w\right) = (u, v, w).$$

This finishes the proof.

2. (a) Let $f := \varphi_\beta^{-1} \circ \varphi_\alpha$ and $(v_1, v_2) = f(u_1, u_2)$. Since $v_1 = \frac{u_1}{u_1^2+u_2^2}$ and $v_2 = \frac{u_2}{u_1^2+u_2^2}$, we have

$$\begin{aligned} f^*dv_1 &= \frac{1}{(u_1^2 + u_2^2)^2}((u_2^2 - u_1^2)du_1 - 2u_1u_2du_2), \\ f^*dv_2 &= \frac{1}{(u_1^2 + u_2^2)^2}(-2u_1u_2du_1 + (u_1^2 - u_2^2)du_2). \end{aligned}$$

This implies that

$$f^*\omega_\beta = -\frac{(u_1^2+u_2^2)^2}{u_1^2}\times$$
$$\left(\frac{u_1}{(u_1^2+u_2^2)^3}((u_2^2-u_1^2)du_1 - 2u_1u_2du_2) + \frac{u_2}{(u_1^2+u_2^2)^3}(-2u_1u_2du_1 + (u_1^2-u_2^2)du_2)\right)$$
$$= -\frac{1}{u_1^2(u_1^2+u_2^2)}((u_1(u_2^2-u_1^2)-2u_1u_2^2)du_1 + (-2u_1^2u_2+u_2(u_1^2-u_2^2)du_2)$$
$$= -\frac{1}{u_1^2(u_1^2+u_2^2)}((-u_1^3-u_1u_2^2)du_1 + (-u_1^2u_2-u_2^3)du_2)$$
$$= -\frac{1}{u_1^2}(-u_1du_1 - u_2du_2) = \omega_\alpha,$$

i.e., the compatibility condition is satisfied and the pair $\{\omega_\alpha, \omega_\beta\}$ corresponds to a uniquely determined global 1-form on S^2.

(b) Since

$$(x,y,z) = \varphi_\alpha(u_1,u_2) = \frac{1}{u_1^2+u_2^2+1}(2u_1, 2u_2, u_1^2+u_2^2-1)$$
$$= \varphi_\beta(v_1,v_2) = \frac{1}{v_1^2+v_2^2+1}(2v_1, 2v_2, 1-v_1^2-v_2^2),$$

we have

$$\begin{aligned}
\varphi_\alpha^* dx &= \frac{2}{(u_1^2+u_2^2+1)^2}((1-u_1^2+u_2^2)du_1 - 2u_1u_2du_2),\\
\varphi_\alpha^* dz &= \frac{4}{(u_1^2+u_2^2+1)^2}(u_1du_1+u_2du_2),\\
\varphi_\alpha^*\left(\frac{-2}{x^3}dx\wedge dz\right) &= \frac{-2}{u_1^3(u_1^2+u_2^2+1)}((1-u_1^2+u_2^2)du_1 - 2u_1u_2du_2)\wedge(u_1du_1+u_2du_2)\\
&= -\frac{2u_2}{u_1^3}du_1\wedge du_2 = d\omega_\alpha,\\
\varphi_\beta^* dx &= \frac{2}{(v_1^2+v_2^2+1)^2}((1-v_1^2+v_2^2)dv_1 - 2v_1v_2dv_2),\\
\varphi_\beta^* dz &= -\frac{4}{(v_1^2+v_2^2+1)^2}(v_1dv_1+v_2dv_2),\\
\varphi_\beta^*\left(\frac{-2}{x^3}dx\wedge dz\right) &= \frac{2}{v_1^3(v_1^2+v_2^2+1)}((1-v_1^2+v_2^2)dv_1 - 2v_1v_2dv_2)\wedge(v_1dv_1+v_2dv_2)\\
&= \frac{2v_2}{v_1^3}dv_1\wedge dv_2 = d\omega_\beta.
\end{aligned}$$

3. (a) We have

$$v \times w = \begin{pmatrix} yc - zb & za - xc & xb - ya \end{pmatrix}^\top.$$

Since v, w are linear independent, the three vectors $v \times w, v, w$ form a basis of $\mathbb{R}^3$ and

$$\det(v \times w, v, w) = \det \begin{pmatrix} yc - zb & x & a \\ za - xc & y & b \\ xb - ya & z & c \end{pmatrix}$$

$$= (yc-zb)yc+(za-xc)za+(xb-ya)xb-(xb-ya)ya-(za-xc)xc-(yc-zb)zb$$

$$= (za - xc)^2 + (yc - zb)^2 + (xb - ya)^2 \geq 0.$$

Since we deal with a basis, we therefore must have $\det(v \times w, v, w) > 0$, i.e., the transition matrix expression $v \times w, v, w$ in terms of e_1, e_2, e_3 has positive determinant.

(b) Let $(x, y, z) = \varphi(s, t)$. Since $-1 < s < 1$, we have $2 + s > 0$ and $\sqrt{x^2 + y^2} = 2 + s$, i.e.,

$$(\sqrt{x^2 + y^2} - 2)^2 = s^2 = z.$$

This shows that with $f(x, y, z) = (\sqrt{x^2 + y^2} - 2)^2 - z$ we have $f(\varphi(s, t)) = 0$. Since, for all (x, y, z) in the domain of f, $Df(x, y, z) = (*, *, -1) \neq 0$, every value is a regular value of f.

(c) Notice that $p = (2, 0, 0) = \varphi_2(0, 0)$ and

$$\frac{\partial \varphi_2}{\partial s}(s, t) = (\cos t, \sin t, 2s), \quad \frac{\partial \varphi_2}{\partial t}(s, t) = (-(2+s)\sin t, (2+s)\cos t, 0).$$

We have, in particular,

$$\frac{\partial \varphi_2}{\partial s}(0, 0) = e_1, \quad \frac{\partial \varphi_2}{\partial t}(0, 0) = 2e_2,$$

and e_3 is orthogonal to $\text{span}(e_1, e_2) = T_pM$. Moreover, $e_3, e_1, 2e_2$ carries the same orientation as e_1, e_2, e_3, therefore e_3 is a positively oriented unit normal vector at $p \in M$ with respect to $\{\varphi_1, \varphi_2\}$.