Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 16 29.2.2012

1. (a) We have
$$d\omega = 3\,dx \wedge dy \wedge dz.$$

(b) An almost coordinate chart of E is given by

$$\varphi : V := (0,2\pi)\times(-\pi/2,\pi/2) \to U \subset E, \quad \varphi(\alpha,\beta) = (a\cos\alpha\cos\beta, b\sin\alpha\cos\beta, c\sin\beta).$$

The points, which are not reached by this parametrisation form a smooth curve connecting south and north pole of the ellipse. This is a set of measure zero. Then we have

$$\begin{aligned}
w_2 := \frac{\partial\varphi}{\partial\alpha}(\alpha,\beta) &= (-a\sin\alpha\cos\beta, b\cos\alpha\cos\beta, 0)^\top,\\
w_3 := \frac{\partial\varphi}{\partial\beta}(\alpha,\beta) &= (-a\cos\alpha\sin\beta, -b\sin\alpha\sin\beta, c\cos\beta)^\top,\\
w_1 := \frac{\partial\varphi}{\partial\alpha}(\alpha,\beta)\times\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) &= \left(bc\cos\alpha\cos^2\beta, ac\sin\alpha\cos^2\beta, ab\sin\alpha\cos\beta\right)^\top.
\end{aligned}$$

By construction w_1, w_2, w_3 have the same orientation as e_1, e_2, e_3 and at $\varphi(\pi, 0) = (-a, 0, 0)$ we have $w_1 = (-bc, 0, 0)$, so that the outer unit normal vector is positively oriented with respect to the orientation induced by this coordinate chart.

We have
$$\int_E \omega = \int_U \omega = \int_V \varphi^*\omega,$$
and

$$\begin{aligned}
\varphi^*\omega &= a\cos\alpha\cos\beta\, d(b\sin\alpha\cos\beta)\wedge\, d(c\sin\beta) -\\
&-b\sin\alpha\cos\beta\, d(a\cos\alpha\cos\beta)\wedge d(c\sin\beta) + c\sin\beta\, d(a\cos\alpha\cos\beta)\wedge d(b\sin\alpha\cos\beta) =\\
&= a\cos\alpha\cos\beta(b\cos\alpha\cos\beta d\alpha - b\sin\alpha\sin\beta d\beta)\wedge c\cos\beta d\beta -\\
&- b\sin\alpha\cos\beta(-a\sin\alpha\cos\beta d\alpha - a\cos\alpha\sin\beta d\beta)\wedge c\cos\beta d\beta +\\
&+ c\sin\beta(-a\sin\alpha\cos\beta d\alpha - a\cos\alpha\sin\beta d\beta)\wedge(b\cos\alpha\cos\beta d\alpha - b\sin\alpha\sin\beta d\beta) =\\
&= abc(\cos^2\alpha\cos^3\beta + \sin^2\alpha\cos^3\beta + \sin^2\alpha\sin^2\beta\cos\beta + \cos^2\alpha\sin^2\beta\cos\beta)\, d\alpha\wedge d\beta =\\
&= abc(\cos^3\beta + \sin^2\beta\cos\beta)\, d\alpha\wedge d\beta = abc\cos\beta\, d\alpha\wedge d\beta.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_E \omega = \int_V \varphi^*\omega = \int_{(0,2\pi)}\int_{(-\pi/2,\pi/2)} abc\cos\beta\, d\beta d\alpha &=\\
= 2\pi abc \int_{(-\pi/2,\pi/2)} \cos\beta\, d\beta &= 4\pi abc.
\end{aligned}$$

2. We use in our arguments the abbreviations $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_{k-1}$. Note also that the definition of i_t implies $Di_t(x)(v) = (0, v)$ and

$$dt(Di_t(x)(v)) = dt(0, v) = 0, \tag{1}$$

since t is the first coordinate in $(t, x_1, \dots, x_n)$.

(a) By linearity, it suffices to prove the formula in (a) only for $\eta = f_I dx_I$ and for $\eta = g_J dt \wedge dx_J$.

Case $\eta = f_I dx_I$: Then $I\eta = 0$, by definition of I, and $d(I\eta) = 0$. On the other hand, we have

$$d\eta = \frac{\partial f_I}{\partial t} dt \wedge dx_I + \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I,$$

which implies

$$\begin{aligned} (I(d\eta))_x &= \int_0^1 \frac{\partial f_I}{\partial t} \underbrace{dx_I(Di_t(x)\cdot, \dots, Di_t(x)\cdot)}_{dx_I(\cdot,\dots,\cdot),\ \text{where } dx_I \in \Omega^k(U)} \\ &= (f_I(1, x) - f_I(0, x)) \cdot dx_I(\cdot, \dots, \cdot). \end{aligned}$$

Then

$$d(I\eta)_x + I(d\eta)_x = (f_I(1, x) - f_I(0, x)) \cdot dx_I$$

and

$$i_1^*\eta - i_0^*\eta = f_I(1, \cdot)dx_I - f_I(0, \cdot)dx_I.$$

Case $\eta = g_j dt \wedge dx_j$: Then $d\eta = -\sum_{j=1}^{n} \frac{\partial g_J}{\partial x_j} dt \wedge dx_j \wedge dx_J$ and

$$(I\eta)_x = \int_0^1 g_J(t, x) \underbrace{dx_J(Di_t(x)\cdot, dots, Di_t(x)\cdot)}_{dx_J(\cdot,\dots,\cdot),\ \text{where } dx_I \in \Omega^k(U)} dt,$$

and

$$d(I\eta)_x = \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial g_J}{\partial x_j}(t, x) dt \right) \underbrace{dx_i \wedge dx_J}_{\in \Omega^k(U)},$$

and

$$\begin{aligned} I(d\eta)_x &= -\int_0^1 \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_j}(t, x) \underbrace{dx_j \wedge dx_J(Di_t(x)\cdot, \dots, Di_t(x)\cdot) dt}_{dx_j \wedge dx_J(\cdot,\dots,\cdot),\ \text{where } dx_j \ \wedge dx_j \in \Omega^k(U)} \\ &= -\sum_{j=1}^{n} \left(\int_0^1 \frac{\partial g_J}{\partial x_j}(t, x) dt \right) dx_j \wedge dx_J(\cdot, \dots, \cdot). \end{aligned}$$

This implies that

$$d(I\eta)_x + I(d\eta)_x = 0.$$

On the other hand, we have $i_1^*\eta - i_0^*\eta = 0$ since

$$(i_t^*\eta)_x(v_1,\dots,v_k) = g_J(t,x)\underbrace{dt \wedge dx_J(Di_t(x)v_1,\dots,Di_t(x)v_k)}_{=0,\text{ because of (1)}}.$$

This shows also in this case that

$$d(I\eta)_x + I(d\eta)_x = i_1^*\eta - i_0^*\eta.$$

(b) We have $H\circ i_1(x) = H(1,x) = x$ and $H\circ i_0(x) = H(0,x) = p$. Note also that if F is constant, then $DF = 0$ and, therefore,

$$(F^*\omega)_x(v_1,\dots,v_k) = \omega_{F(x)}(\underbrace{DF(x)v_1}_{=0},\dots,\underbrace{DF(x)v_k}_{=0}) = 0,$$

i.e., $F^*\omega = 0$. Let $\alpha = I(H^*\omega) \in \Omega^{k-1}(U)$. We have, by (a)

$$i_1^*H^*\omega - i_0^*H^*\omega = d(\underbrace{I(H^*\omega)}_{=\alpha}) + I(d(H^*\omega)),$$

and therefore

$$\omega - 0 = d\alpha + I(d(H^*\omega)) = d\alpha + I(H^*(\underbrace{d\omega}_{=0,\ \omega\text{ closed}})) = d\alpha,$$

i.e., ω is exact.