Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 2 17.10.2011

1. (a) Choose a sequence $x_j \in K_j$. This sequence is a Cauchy sequence, since for $\epsilon > 0$ there exists a j such that $\text{diam}(K_j) < \epsilon$. Then, since $K_l \subset K_j$ for all $l \geq j$, we have for all $n, m \geq j$: $x_n, x_m \in K_j$ and, therefore,

$$d(x_n, x_m) \leq \text{diam}(K_j) < \epsilon.$$

Since (M, d) is complete, x_n is convergent: $x_n \to x \in M$. We show that $x \in \bigcap_{j=1}^{\infty} K_j$. This is true if we have $x \in K_j$ for all j. Since $x_n \to x$ and $x_n \in K_j$ for all $n \geq j$, and K_j is closed, we conclude from Proposition 1.15: $x \in K_j$. Finally, we show that x is the only point in the intersection. Assume $x, y \in \bigcap K_j$ with $x \neq y$. Then $d(x, y) = \epsilon > 0$. Choose n such that $\text{diam}(K_n) < \epsilon$. Since $x, y \in K_n$, we must have $d(x, y) < \epsilon$, a contradiction.

(b) Let $U_j = (0, 1/j) \subset \mathbb{R}$. Then the U_j are nested, $\text{diam}(U_j) = 1/j \to 0$, and $\bigcap_{j=1}^{\infty} U_j = \emptyset$.

2. Assume that $f_n \to f \in B([a, b])$. Let $x \in [a, b]$ and $\epsilon > 0$. Then there exists n_0 such that $d(f_n, f) < \epsilon/3$ for all $n \geq n_0$. In particular, $d(f_{n_0}, f) < \epsilon/3$. Since f_{n_0} is continuous, there is a $\delta > 0$ such that $|f_{n_0}(y) - f_{n_0}(x)| < \epsilon/3$ for all y with $|y - x| < \delta$. This implies that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| \\ &< d(f, f_{n_0}) + \epsilon/3 + d(f, f_{n_0}) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

for all y with $|y - x| < \delta$. This means that f is continuous at x.

3. Let $(A_\alpha)_{\alpha \in I}$ be a family of open sets. Let $x \in \bigcup A_\alpha$. Then there exists $\alpha_0 \in I$ with $x \in A_{\alpha_0}$. Since A_{α_0} is open, there exists an open ball $U_R(x) \subset A_{\alpha_0}$ for $R > 0$ small enough. This implies $U_R(x) \subset \bigcup A_\alpha$, i.e., $\bigcup A_\alpha$ is open.

4. We obviously have $\|h_j\| = 1$, since h_j is the normalisation of g_j. We will prove by induction that for all $n \in \mathbb{N}_0$: h_n is orthogonal to any h_k with $k < n$. For $n = 0$ there is nothing to prove. Assume the statement is true for all integer values below n. Then, for $k < n$, we have

$$\begin{aligned} \langle g_n, h_k \rangle &= \langle f_n - \sum_{l=0}^{n-1} \langle f_n, h_l \rangle h_l, h_k \rangle \\ &= \langle f_n, h_k \rangle - \sum_{l=0}^{n-1} \langle f_n, h_l \rangle \cdot \langle h_l, h_k \rangle. \end{aligned}$$

Note in the formula above, we have $\langle h_l, h_k \rangle = \delta_{lk}$, by the induction hypothesis. This yields

$$\langle g_n, h_k \rangle = \langle f_n, h_k \rangle - \langle f_n, h_k \rangle = 0,$$

i.e., g_n is orthogonal to h_k. Since h_n is just the normalisation of g_n, the same holds for h_n. This finishes the induction step.

The procedure yields:

$$\begin{aligned} h_0(x) &= 1, \\ h_1(x) &= x - \langle x, 1 \rangle 1 = x - \frac{1}{2}, \\ h_2(x) &= x^2 - \langle x^2, x - \frac{1}{2} \rangle (x - \frac{1}{2}) - \langle x^2, 1 \rangle 1 \\ &= x^2 - \frac{5}{6}(x - \frac{1}{2}) - \frac{1}{3} = x^2 - \frac{5}{6}x + \frac{1}{12}. \end{aligned}$$

5. The elements $\mathbf{s}_j \in B_1(\mathbf{0})$ obviously satisfy $d(\mathbf{s}_j, \mathbf{s}_k) = 2$. Assume there would be finitely many balls $U_{1/2}(\mathbf{x}_1), \ldots, U_{1/2}(\mathbf{x}_k)$ covering $B_1(\mathbf{0})$. If $\mathbf{s}_i \in U_{1/2}(\mathbf{x}_j)$, then $U_{1/2}(\mathbf{x}_j) \subset U_1(\mathbf{s}_i)$, by triangle inequality. But this means that no other $\mathbf{s}_l$ can lie in $U_{1/2}(\mathbf{x}_j)$, since all $\mathbf{s}_l$ have distance two from each other. This shows that every ball $U_{1/2}(\mathbf{x}_j)$ contains at most one of the points $\mathbf{s}_l \in B_1(\mathbf{0})$. Since there are infinitely many $\mathbf{s}_l$, this leads to a contradiction.