Analysis III/IV (Math 3011, Math 4201)

Solutions to Exercise Sheet 9 7.12.2011

1. Note that
$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \frac{\partial f}{\partial x_2}\, dx_2 + \frac{\partial f}{\partial x_3}\, dx_3.$$
On the other hand, we have
$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right),$$
i.e.,
$$\omega_{\nabla f} = \frac{\partial f}{\partial x_1}\, dx_1 + \frac{\partial f}{\partial x_2}\, dx_2 + \frac{\partial f}{\partial x_3}\, dx_3.$$
Next, note for a vector field $F = (f_1, f_2, f_3)$ we have
$$\operatorname{curl} F = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right),$$
i.e.,
$$\eta_{\operatorname{curl} F} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2.$$
On the other hand, since $\omega_F = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$,
$$d\omega_F = \left(\frac{\partial f_1}{\partial x_2} dx_2 + \frac{\partial f_1}{\partial x_3} dx_3\right) \wedge dx_1 + \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_3} dx_3\right) \wedge dx_2 + \left(\frac{\partial f_3}{\partial x_1} dx_1 + \frac{\partial f_3}{\partial x_2} dx_2\right) \wedge dx_3,$$
which shows $\eta_{\operatorname{curl} F} = d\omega_F$, after rearranging the latter expression and using the fact that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Finally, for $F = (f_1, f_2, f_3)$ and $\eta_F = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$, we obtain
$$d\eta_F = \frac{\partial f_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial f_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial f_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2,$$
i.e.,
$$d\eta_F = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 = \operatorname{div} F dx_1 \wedge dx_2 \wedge dx_3.$$
The above calculations show that the following diagram commutes:
$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathcal{X}(\mathbb{R}^3) & \xrightarrow{\text{curl}} & \mathcal{X}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\
\downarrow{\Phi_0} & & \downarrow{\Phi_1} & & \downarrow{\Phi_2} & & \downarrow{\Phi_3} \\
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$
Here, $\mathcal{X}(U)$ denotes the space of smooth vector fields on $U \subset \mathbb{R}^n$, and the vertical maps Φ_i are **bijective maps** between functions/vector fields and differential forms and defined as follows:
$$\Phi_0(f) = f, \quad \Phi_1(F) = \omega_F, \quad \Phi_2(F) = \eta_F, \quad \Phi_3(f) = f dx_1 \wedge dx_2 \wedge dx_3.$$

Since the vertical maps are bijective, we see that $d^2 = 0$ translates into $\text{curl} \circ \nabla = 0$ and $\text{div} \circ \text{curl} = 0$.

2. (a) We have

$$\left| \int_c \omega \right| = \left| \int_a^b \omega_{c(t)}(c'(t))dt \right| \leq \int_a^b \left| \langle F_\omega(c(t)), c'(t) \rangle \right| dt \leq$$
$$\leq \int_a^b \|F_\omega(c(t))\| \cdot \|c'(t)\| dt \leq M \int_a^b \|c'(t)\| dt = ML(c).$$

(b) According to Proposition 5.13, we only habe to prove that we have $\int_c \omega = 0$ for all closed curves $c : [a, b] \to \mathbb{R}^n - 0$. Choose $M > 0$ and $r > 0$ such that $\|F_\omega(x)\| \leq M$ for all $\|x\| \leq r$. Let $\epsilon > 0$ be arbitrary. We consider the free homotopy $G : [a, b] \times [\epsilon, 1] \to \mathbb{R}^n - 0$, defined by $G(t, s) = s \cdot c(t)$. Since G is a free homotopy and ω is closed, we conclude that

$$\int_c \omega = \int_{c_\epsilon} \omega,$$

by Corollary 6.13. Note also that $L(c_\epsilon) = \epsilon \cdot L(c)$, since $c'_\epsilon(t) = \epsilon c'(t)$. This implies with (a) that

$$\left| \int_c \omega \right| = \left| \int_{c_\epsilon} \omega \right| \leq M \cdot L(c_\epsilon) = \epsilon \cdot M \cdot L(c).$$

Since $\epsilon > 0$ was arbitrary, we must have $\int_c \omega = 0$. This is what we wanted to show.

(c) Note that $F_\omega(x, y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ and

$$\|F_\omega(0, y)\| = \frac{1}{|y|}.$$

Note that $1/|y|$ is not bounded for any disk of centre 0, so we cannot apply (b) in this case.

3. (a) Since $c'_x(t) = x = (x_1, x_2)$, we have

$$f(x) = \int_{c_x} \omega = \int_0^1 \omega_{c_x(t)}(c'_x(t))dt = \int_0^1 f_1(c_x(t))dx_1(c'_x(t)) + f_2(c_x(t))dx_2(c'_x(t)) =$$
$$\int_0^1 f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2 dt.$$

(b) Since $\omega = f_1 dx_1 + f_2 dx_2$ is closed, we have $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$. Using this, we obtain

$$\begin{aligned}
\frac{\partial f}{\partial x_1}(x) &= \int_0^1 \frac{\partial}{\partial x_1}\left(f_1(tx_1,tx_2)x_1 + f_2(tx_1,tx_2)x_2\right) dt \\
&= \int_0^1 \left(t\frac{f_1}{\partial x_1}(tx_1,tx_2)x_1 + f_1(tx_1,tx_2) + t\frac{f_2}{\partial x_1}(tx_1,tx_2)x_2\right) dt \\
&= \int_0^1 \left(t\frac{f_1}{\partial x_1}(tx_1,tx_2)x_1 + t\frac{f_1}{\partial x_2}(tx_1,tx_2)x_2\right) dt + \int_0^1 f_1(tx_1,tx_2)dt \\
&= \int_0^1 t\langle \nabla f(c_x(t)), c_x'(t)\rangle dt + \int_0^1 f_1(c_x(t))dt \\
&= \int_0^1 tDf_1(c_x(t))(c_x'(t))dt + \int_0^1 f_1(c_x(t))dt \\
&= \int_0^1 t(f_1\circ c_x)'(t) + f_1\circ c_x(t)dt,
\end{aligned}$$

where in the last step we applied the chain rule. Partial integration yields

$$\frac{\partial f}{\partial x_1}(x) = [tf_1\circ c_x(t)]_0^1 - \int_0^1 f_1\circ c_x(t)dt + \int_0^1 f_1\circ c_x(t)dt = 1\cdot f_1(x) - 0\cdot f_1(0) = f_1(x).$$

Similarly, one shows $\frac{\partial f}{\partial x_2}(x) = f_2(x)$, and we conclude that

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 = f_1dx_1 + f_2dx_2 = \omega,$$

i.e., ω is exact.