Some lattice subgroups of $PU(2,1)$

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Newcastle University, June 5-11 2016.

Motivation: A **fake projective plane** is a compact complex surface with the same Betti numbers as the (nonfake) projective plane $\mathbb{P}^2(\mathbb{C})$, namely 1, 0, 1, 0, 1, but which is not homeomorphic to $\mathbb{P}^2(\mathbb{C})$.

It is known that any fpp has the form $B(\mathbb{C}^2)/\Pi$, where Π is a cocompact discrete subgroup of $PU(2,1)$ such that

(a) Π is torsion-free,

(b) $\mu(PU(2,1)/\Pi) = 1$,

(c) $\Pi/[\Pi,\Pi]$ is finite, and

(d) Π is arithmetic.

By (d), Π must be contained as a subgroup of finite index, N say, in a maximal arithmetic subgroup, $\bar{\Gamma}$ say, of $PU(2,1)$. Thus

$$\mu(PU(2,1)/\bar{\Gamma}) = \frac{1}{N}.$$

Prasad and Yeung showed that this condition on a maximal arithmetic subgroup $\bar{\Gamma}$ is extremely restrictive. There is a list of fewer than 100 possibilities for $\bar{\Gamma}$, which they wrote down not quite explicitly. Many of these can not give an fpp, because they have no torsion-free subgroup of index N. They use:

Lemma 1. Suppose that Π is a torsion-free subgroup of finite index in a group $\bar{\Gamma}$. Let K be a finite subroup of $\bar{\Gamma}$. Then $|K|$ divides $[\bar{\Gamma} : \Pi]$.

In particular, if $\bar{\Gamma}$ has an element of order n, and n does not divide N, then $\bar{\Gamma}$ contains no torsion-free subgroup of index N.

Each of these $\bar{\Gamma}$'s is associated with a pair (k, ℓ) of fields, where k is either $\mathbb{Q}$ or a real quadratic extension of $\mathbb{Q}$, and ℓ a complex quadratic extension of k, and with a central simple algebra (either a division algebra $\mathcal{D}$ of dimension 9 over ℓ or the matrix algebra $M_{3\times 3}(\ell)$) and an hermitian form (on either $\mathcal{D}$ or ℓ^3). Prasad and Yeung found about 20 new fpp's by looking at these $\bar{\Gamma}$'s. Tim Steger and I completed this work by going through all the possibilities for $\bar{\Gamma}$, and finding all the Π's of index N in $\bar{\Gamma}$ which are torsion-free with $\Pi/[\Pi, \Pi]$ finite. We showed that

- there are precisely 50 distinct fpp's,

- all of these come from $\bar{\Gamma}$'s associated with a division algebra.

There were altogether 13 $\bar{\Gamma}$'s associated with six pairs (k,ℓ) of fields and the matrix algebra $M_{3\times 3}(\ell)$. Today I am mostly talking about how we showed that no fpp's arise in these cases.

- the methods are similar to those used in the division algebra case,

- there are some new methods which simplify these cases,

- For just one of these 13 $\bar{\Gamma}$'s, a torsion-free Π of the right index does exist, but $\Pi/[\Pi,\Pi]$ is infinite. The surface $B(\mathbb{C}^2)/\Pi$ has recently been studied by me in joint work with Yeung and Koziarz.

The action of $PU(2,1)$ on $B(\mathbb{C}^2)$. For

$$F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

define

$$U(2,1) = \{g \in M_{3\times 3}(\mathbb{C}) : g^* F_0 g = F_0\},$$

$$PU(2,1) = U(2,1)/Z \quad \text{for } Z = \{tI : |t| = 1\},$$

$$SU(2,1) = \{g \in U(2,1) : \det(g) = 1\}.$$

There are natural maps $SU(2,1) \to U(2,1) \to PU(2,1)$.

The action of $PU(2,1)$ on $B(\mathbb{C}^2) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$:

$$(gZ).(z_1, z_2) = (w_1, w_2) \quad \Leftrightarrow \quad g \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix} \quad \text{for some } \lambda.$$

This action preserves the hyperbolic metric d on $B(\mathbb{C}^2)$

$$\cosh^2(d(z,w)) = \frac{|1 - \langle z, w\rangle|^2}{(1-|z|^2)(1-|w|^2)}.$$

where $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ and $|z|^2 = |z_1|^2 + |z_2|^2$.

For the origin $0 := (0,0)$ in $B(\mathbb{C}^2)$,

$$g.0 = (g_{13}/g_{33}, g_{23}/g_{33}) \quad \text{and}$$

$$\cosh^2(d(0, g.0)) = |g_{33}|^2$$

because $g^* F_0 g = F_0$ implies that g satisfies the "column 3 condition":

$$|g_{13}|^2 + |g_{23}|^2 = |g_{33}|^2 - 1.$$

The column 3 condition is seen by looking at the $(3,3)$-entry of $g^*Fg - F = 0$. From the $(1,1)$-entry of $gF^{-1}g^* - F^{-1} = 0$, we get

$$|g_{11}|^2 + |g_{12}|^2 = |g_{13}|^2 + 1, \quad \text{“the row 1 condition.”}$$

Using also

$$g^{-1} = \frac{1}{\theta} g^{\mathrm{adj}} = F^{-1} g^* F,$$

where $\theta = \det(g)$, it is easy to prove the following:

Lemma 2. Given five complex numbers g_{11}, g_{12}, g_{13}, g_{23} and g_{33} satisfying the column 3 and row 1 conditions, and given any $\theta \in \mathbb{C}$ with $|\theta| = 1$, there is a unique $g \in U(2,1)$ with the given five entries and with $\det(g) = \theta$.

We give details for just one of the 13 $\bar{\Gamma}$'s: the "$(\mathcal{C}_{11}, \emptyset)$" case.

Let $\ell = \mathbb{Q}(\zeta)$, where ζ is a primitive 12-th root of 1. Then $[\ell : \mathbb{Q}] = 4$, with $\zeta^4 - \zeta^2 + 1 = 0$. This ℓ contains $k = \mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(i)$, because $(\zeta^3)^2 = -1$ and $r^2 = 3$ for $r = \zeta + \zeta^{-1}$. The following F has determinant 1:

$$F = \begin{pmatrix} -r-1 & 1 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and entries which are algebraic integers in k.

- If $r = +\sqrt{3}$, then two eigenvalues of F are negative, and one is positive,

- if $r = -\sqrt{3}$, all three eigenvalues of F are positive.

Let $\mathfrak{o}_\ell$ denote the ring of algebraic integers in ℓ. In this case,

$$\mathfrak{o}_\ell = \mathbb{Z}[\zeta] = \{a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 : \ a_j \in \mathbb{Z} \text{ for each } j\}.$$

Let

$$\bar{\Gamma} = \{g \in M_{3\times 3}(\mathfrak{o}_\ell) : g^*Fg = F\}/\mathcal{Z},$$

where

$$\mathcal{Z} = \{tI : t \in \mathfrak{o}_\ell \text{ and } |t| = 1\} = \{\zeta^\nu I : \nu = 0, \ldots, 11\}.$$

The other 12 $\bar{\Gamma}$'s are defined in the same way, for different k, ℓ and F.

Writing

$$\Delta = \begin{pmatrix} r+1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{r+1} \end{pmatrix},$$

we find that $\Delta^* F_0 \Delta = -(r+1)F$, and so for $\tilde{g} = \Delta g \Delta^{-1}$,

$$g^*Fg = F \quad \text{if and only if} \quad \tilde{g}^* F_0 \tilde{g} = F_0.$$

So

$$g\mathcal{Z} \mapsto \tilde{g}Z$$

embeds $\bar{\Gamma}$ in $PU(2,1)$. Using Prasad's Covolume Formula, we have the following

Fact: For Haar measure on $PU(2,1)$ normalized in a suitable way,

$$\mu(PU(2,1)/\bar{\Gamma}) = \frac{1}{864}.$$

Here is another way of thinking of this. The embedding of $\bar{\Gamma}$ in $PU(2,1)$ gives an action of $\bar{\Gamma}$ on $B(\mathbb{C}^2)$. Let $\mathcal{F}_{\bar{\Gamma}} \subset B(\mathbb{C}^2)$ be a fundamental domain for this action; for example, the Dirichlet fundamental domain

$$\mathcal{F}_{\bar{\Gamma}} = \{z \in B(\mathbb{C}^2) : d(0,z) \leq d(g.0,z) \text{ for all } g \in \bar{\Gamma}\}.$$

Then with suitably normalized hyperbolic volume on $B(\mathbb{C}^2)$,

$$\text{vol}(\mathcal{F}_{\bar{\Gamma}}) = \frac{1}{864}.$$

Question: Does $\bar{\Gamma}$ have a torsion-free subgroup Π of index 864?

If we can find such a subgroup with $\Pi/[\Pi,\Pi]$ is finite, then Π would be the fundamental group of a fake projective plane.

Answer: Up to conjugacy, there is a unique torsion-free subgroup of index 864, but $\Pi/[\Pi,\Pi] = \mathbb{Z}^2$.

The compact complex surface $B(\mathbb{C}^2)/\Pi$ is a new and interesting surface, not a fake projective plane.

To give the above answer, we need to find lots of elements of $\bar{\Gamma}$.

There are column 3 and row 1 conditions on the $g = (g_{ij})$ satisfying $g^*Fg = F$:

$$|g_{13}|^2 + |g_{13} - (r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1),$$

and

$$|g_{11}|^2 + |g_{11} + (r+1)g_{12}|^2 = (r+1)|g_{13}|^2 + 2.$$

Lemma 3. Given five numbers g_{11}, g_{12}, g_{13}, g_{23} and g_{33} in ℓ satisfying these column 3 and row 1 conditions, and given any $\theta \in \ell$ with $|\theta| = 1$, there is a unique $g \in M_{3\times 3}(\ell)$ with $g^*Fg = F$, the given five entries, and with $\det(g) = \theta$.

Lemma 4. Let $\alpha \in \mathfrak{o}_\ell$. Then we can write

$$|\alpha|^2 = P(\alpha) + Q(\alpha) r,$$

where

- $P(\alpha), Q(\alpha) \in \mathbb{Z}$,
- $P(\alpha) \geq 0$, with equality iff $\alpha = 0$,
- $|Q(\alpha)| \leq \frac{1}{r} P(\alpha)$.

Writing $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, we have

$$P(\alpha) = a_0^2 + a_0a_2 + a_1^2 + a_1a_3 + a_2^2 + a_3^2 \geq \frac{1}{2}\Big(a_0^2 + a_1^2 + a_2^2 + a_3^2\Big),$$

and

$$Q(\alpha) = a_0a_1 + a_1a_2 + a_2a_3.$$

With these formulas, we can write down a list of possible values of $(P(\alpha), Q(\alpha))$ with $P(\alpha) \leq B$ for a given bound B. This list starts

$$(0,0),\ (1,0),\ (2,-1),\ (2,0),\ (2,1),\ (3,0),\ (4,-2),\ (4,-1),\ (4,0),\ \ldots$$

We can also identify

$$\{\alpha \in \mathfrak{o}_\ell : P(\alpha) = p \text{ and } Q(\alpha) = q\}$$

for each (p,q) in the list.

The next step in finding elements of $\bar{\Gamma}$ is to identify

$$K = \{g \in \bar{\Gamma} : g.0 = 0\}.$$

(we are usually just going to write g, not $g\mathcal{Z}$, for elements of $\bar{\Gamma}$).

Now $g.0 = 0$ iff $\tilde{g}.0 = 0$ iff $\tilde{g}_{13} = \tilde{g}_{23} = 0$, which holds iff $g_{13} = 0$ and $g_{23} = 0$. Then $|g_{33}| = 1$ and wlog $g_{33} = 1$. So wlog g has the form

$$\begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The g_{ij}'s here must be in $\mathfrak{o}_\ell$. The entries g_{11} and g_{12} must satisfy

$$|g_{11}|^2 + |g_{11} + (r+1)g_{12}|^2 = 2,$$

which is just the row 1 condition in the case $g_{13} = 0$.

This equation has the form $|\alpha|^2 + |\beta|^2 = 2$, where $\alpha, \beta \in \mathfrak{o}_\ell$. We must have $P(\alpha) + P(\beta) = 2$ and $Q(\alpha) + Q(\beta) = 0$. We read from our lists of (p, q)'s, etc, the possibilities for α and β. For each such α and β, we solve for g_{11} and g_{12}. Then g_{11}, g_{12}, 0, 0, 1 satisfy the row 1 and column 3 conditions, and so g_{21} and g_{22} are determined by g_{11}, g_{12} and the choice of $\theta = \det(g)$. Running through the possibilities for α and β, and for θ, and checking when the g_{ij}'s are in $\mathfrak{o}_\ell$, we get:

Lemma 5. There are 288 elements in K, which is generated by $u\mathcal{Z}$ and $v\mathcal{Z}$ for the matrices

$$u = \begin{pmatrix} \zeta^3+\zeta^2-\zeta & 1-\zeta & 0 \\ \zeta^3+\zeta^2-1 & \zeta-\zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \zeta^3 & 0 & 0 \\ \zeta^3+\zeta^2-\zeta-1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These satisfy

$$u^3 = I,\ v^4 = I,\ \text{and } (uv)^2 = (vu)^2,$$

and these generators and relations give a presentation for K.

For five of the thirteen $\bar{\Gamma}$'s we need to look at, the calculation of K is enough to eliminate there being a torsion-free subgroup in $\bar{\Gamma}$ of the right index.

The next step in our search for elements of $\bar{\Gamma}$ is to find $g \in \bar{\Gamma}$ for which $d(0, g.0)$ is small. Since

$$\cosh^2(d(0, g.0)) = \cosh^2(d(0, \tilde{g}.0)) = |\tilde{g}_{33}|^2 = |g_{33}|^2,$$

we are just looking for $g \in \bar{\Gamma}$ with $|g_{33}| > 1$ but small.

The column 3 condition for $g \in \bar{\Gamma}$ is of the form

$$|\alpha|^2 + |\beta|^2 = (r-1)(|\gamma|^2 - 1), \qquad (\dagger)$$

with $\alpha, \beta, \gamma \in \mathfrak{o}_\ell$, $\gamma = g_{33}$, and with $g.0 = 0$ if and only if $(\alpha, \beta) = (0, 0)$.

Lemma 6. If $\alpha, \beta, \gamma \in \mathfrak{o}_\ell$ satisfy $(\dagger)$, and if $(\alpha, \beta) \neq (0, 0)$, then

$$|\gamma|^2 \geq 2 + r, \quad \text{with equality iff} \quad |\alpha|^2 + |\beta|^2 = 2.$$

It is now easy to find an element g of $\bar{\Gamma}$ with $|g_{33}|^2 = 2 + r$. We find all possible $g_{11}, g_{12}, g_{13}, g_{23}, g_{33} \in \mathfrak{o}_\ell$ satisfying the column 3 and row 1 conditions and $|g_{33}|^2 = 2 + r$, and apply Lemma 3 above.

We find that $\{g \in \bar{\Gamma} : |g_{33}|^2 = 2 + r\} = KbK$ for

$$b = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3 - \zeta^2 + 2\zeta + 2 & \zeta^3 + \zeta^2 - \zeta - 1 & -\zeta^3 - \zeta^2 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & -\zeta^3 + \zeta + 1 \end{pmatrix}.$$

Lemma 7. The elements u, v and b generate $\bar{\Gamma}$.

The corresponding calculations are enough to eliminate three more of the 13 $\bar{\Gamma}$'s. We are able to find elements $g \in \bar{\Gamma}$ of finite order n which does not divide the required $N = [\bar{\Gamma} : \Pi]$, and apply Lemma 1.

For the remaining 5 cases, we find a presentation of the $\bar{\Gamma}$'s. In our example case,

Proposition. The generators u, v and b, together with the relations

$$u^3 = v^4 = b^3 = 1,\ (uv)^2 = (vu)^2,\ vb = bv,\ (buv)^3 = (buvu)^2 v = 1,$$

form a presentation for $\bar{\Gamma}$.

This particular $\bar{\Gamma}$ was known by various experts to be isomorphic to one of the Deligne-Mostow groups, which have nice presentations (see John Parker [2009]). Using this (and some help from John Parker), we could simplify a little the earlier presentation we had from our methods.

For all the $\bar{\Gamma}$'s we were able to get a presentation as follows.

Lemma 8. If $\alpha, \beta, \gamma \in \mathfrak{o}_\ell$ satisfy (†), and if $(\alpha, \beta) \neq (0, 0)$, then

$$0 \leq Q(\gamma) \leq \frac{1}{r} P(\gamma) \leq Q(\gamma) + \frac{1}{r}.$$

Lemma 8 is useful for seeing explicitly the discreteness of the set of distances $d(0, g.0)$, $g \in \bar{\Gamma}$. It implies that

$$2P(g_{33}) - 1 \leq |g_{33}|^2 \leq 2P(g_{33}).$$

Caution: The set $\{|\alpha|^2 : \alpha \in \mathfrak{o}_\ell\}$ is *not* a discrete subset of $\mathbb{R}$. For example,

$$|\zeta - 1|^2 = 2 - r, \quad \text{and so} \quad 0 < |(\zeta - 1)^n|^2 = (2 - \sqrt{3})^n \to 0.$$

Let

$$d_0 = 0 < d_1 < d_2 < \cdots$$

be the distinct values taken by $d(0, g.0)$, $g \in \bar{\Gamma}$. So $\cosh^2(d_n) = p_n + q_n r$ for certain integers p_n and q_n. The first few $p_n + q_n r$'s are:

$$1,\ 2 + r,\ 4 + 2r,\ 6 + 3r,\ 7 + 4r,\ 11 + 6r,\ \ldots$$

We find all possible $g_{11}, g_{12}, g_{13}, g_{23}, g_{33} \in \mathfrak{o}_\ell$ satisfying the column 3 and row 1 conditions and $|g_{33}|^2 = p_n + q_n r$, and then for each $\theta \in \mathfrak{o}_\ell$ such that $|\theta| = 1$, we apply Lemma 3 to form the unique $g \in M_{3\times 3}(\ell)$ with the five specified entries such that $g^* F g = F$ and $\det(g) = \theta$, then test whether the g_{ij}'s are in $\mathfrak{o}_\ell$. In this way, we can form

$$S_n = \{g \in \bar{\Gamma} : d(0, g.0) \le d_n\}.$$

Then

$$K = S_0 \subset S_1 \subset S_1 \subset S_2 \subset \cdots, \quad \text{and} \quad \bigcup_n S_n = \bar{\Gamma}.$$

Form

$$\mathcal{F}_n = \{z \in B(\mathbb{C}^2) : d(0,z) \leq d(g.0, z) \text{ for all } g \in S_n\},$$

$$B(\mathbb{C}^2) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_1 \supset \cdots \quad \text{and} \quad \bigcap_n \mathcal{F}_n = \mathcal{F}_{\bar{\Gamma}}.$$

Let

$$r_n = \max\{d(0,z) : z \in \mathcal{F}_n\} \quad \text{and} \quad r_{\bar{\Gamma}} = \max\{d(0,z) : z \in \mathcal{F}_{\bar{\Gamma}}\}.$$

So

$$\infty = r_0 \geq r_1 \geq r_2 \geq \cdots$$

Lemma 9. If $d_n \geq r_n$, then S_n generates $\bar{\Gamma}$.

Lemma 10. If $d_n \geq 2r_n$, then

(a) $\mathcal{F}_n = \mathcal{F}_{\bar{\Gamma}}$ and $r_n = r_{\bar{\Gamma}}$.

(b) the set S_n of generators, together with the relations $g_1 g_2 g_3 = 1$ which hold for $g_1, g_2, g_3 \in S_n$, form a presentation for $\bar{\Gamma}$.

Lemma 11. For the $(\mathcal{C}_{11}, \emptyset)$ example,

$$r_1 = r_2 = \cdots = \frac{1}{2} d_2 = \frac{1}{2} \cosh^{-1}(1 + \sqrt{3}),$$

so that we take $n = 2$ in Lemmas 9 and 10.

To calculate r_n, we have to maximize $d(0,z)$ subject to the constraints $d(0,z) \leq d(g.0,z)$, $g \in S_n$. Since

$$d(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right), \quad \text{where } |z| = \sqrt{|z_1|^2 + |z_2|^2},$$

this amounts to maximizing

$$|z_1|^2 + |z_2|^2$$

subject to the constraints

$$|g_{31}z_1 + g_{32}z_2 + g_{33}| \geq 1 \quad \text{for all } g \in S_n.$$

While we haven't calculated r_n exactly in most other lattices subgroups in our list of thirteen, we can numerically estimate these numbers with sufficient accuracy to check the condition $d_n \geq 2r_n$.

For finitely presented groups G, Magma and other computer algebra packages have routines for finding subgroups of low index. For the five $\bar{\Gamma}$'s not yet eliminated, the index in question is not low enough for these general routines to work.

Steger and I wrote specialized C-programs to look for torsion-free subgroups of the required index. In the "$(\mathcal{C}_{11}, \emptyset)$" example, this amounted to looking for a permutation B of $\{1, \ldots, 864\}$ with special properties corresponding to the relations satisfied by the generator b of $\bar{\Gamma}$.

This quickly found a torsion-free subgroup Π of $\bar{\Gamma}$ of index 864, but with $\Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$. It took many CPU-days to show that Π was the unique (up to conjugation) torsion-free subgroup of index 864.

If $\Pi \subset \bar{\Gamma}$ is torsion-free and of index 864, then $X = B(\mathbb{C}^2)/\Pi$ is a compact complex surface of Euler-Poincaré characteristic 3. It is *not* a fake projective plane. Sai-Kee Yeung, Vincent Koziarz and I have recently studied geometric properties of X, showing in particular:

Proposition. The Picard number of X is 3. Let $\alpha : X \to T$ be the Albanese map. Then $T \cong \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, where $\omega = e^{2\pi i/3}$, and the genus of the generic fibre of α is 19.

This was shown by studying certain "mirrors" $M_\alpha = \{(z, w) \in B(\mathbb{C}^2) : z = \alpha w\}$ and the groups $\Pi_\alpha = \{\pi \in \Pi : \pi(M_\alpha) = M_\alpha\}$. If $\alpha = 0$, then the generator v of $\bar{\Gamma}$ fixes each point of M_α, and Π_α is a surface group of genus 4. If $\alpha = \zeta^2 - \zeta$, then the generator u of $\bar{\Gamma}$ fixes each point of M_α, and Π_α is a surface group of genus 10.

Where the 13 $\bar{\Gamma}$'s come from.

Here are the (k,ℓ)'s which were not eliminated in [PY]:

name	k	ℓ	defining polynomial for ℓ
$\mathcal{C}_1$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	$\zeta^4+\zeta^3+\zeta^2+\zeta+1$
$\mathcal{C}_3$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{5},i)\cong\mathbb{Q}(z)$	z^4+3z^2+1
$\mathcal{C}_8$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{2},i)\cong\mathbb{Q}(\zeta_8)$	ζ^4+1
$\mathcal{C}_{11}$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{3},i)\cong\mathbb{Q}(\zeta_{12})$	$\zeta^4-\zeta^2+1$
$\mathcal{C}_{18}$	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(\sqrt{6},\zeta_3)\cong\mathbb{Q}(z)$	z^4-2z^2+4
$\mathcal{C}_{21}$	$\mathbb{Q}(\sqrt{33})$	$\mathbb{Q}(\sqrt{33},\zeta_3)\cong\mathbb{Q}(z)$	$z^4-z^3-2z^2-3z+9$

We can define an hermitian form on ℓ^3 by choosing a matrix F, as follows:

We set

$$F = \begin{pmatrix} -x & 0 & 0 \\ 0 & -x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, and

$$F = \begin{pmatrix} -x & 1 & 0 \\ 1 & -2x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, where x is as in the following table:

	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{C}_8$	$\mathcal{C}_{11}$	$\mathcal{C}_{18}$	$\mathcal{C}_{21}$
r^2	5	5	2	3	6	33
x	$(r+1)/2$	$(r+1)/2$	$r+1$	$r+1$	$r+2$	$(r+5)/2$

In each case, $\det(F) = 1$, and all the entries of F are algebraic integers of k.

Each x is positive when r is taken as the positive square root of r^2, and negative when r is taken as the negative square root. There is an algebraic group G defined over k so that

$$G(k) = \{g \in M_{3\times 3}(\ell) : g^* F g = F \text{ and } \det(g) = 1\}.$$

The field k has two archimedean places v^+ and v^- corresponding to the embeddings $k \to \mathbb{R}$ mapping r to the positive and negative square roots of r^2. Taking completions of k, the above sign change of x implies that

$$G(k_{v^+}) \cong SU(2,1) \quad \text{and} \quad G(k_{v^-}) \cong SU(3).$$

If a different choice is made of F, so that the corresponding G behaves in this same way at v^+ and v^-, then the two G's are k-isomorphic.

In our context Π being arithmetic means that there is a "principal arithmetic subgroup"

$$\Lambda = G(k) \cap \prod_{v \in V_f} P_v$$

which is commensurable with $\tilde{\Pi} = \varphi^{-1}(\Pi)$, where $\varphi : SU(2,1) \to PU(2,1)$ is the natural map. Here V_f is the set of non-archimedean places of k, and each P_v is a "parahoric" subgroup of $G(k_v)$. We regard $G(k)$ as a subgroup of $SU(2,1)$ by a suitable conjugation. More exactly, $\Delta^* F_0 \Delta = -xF$ for

$$\Delta = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{x} \end{pmatrix} \quad \text{or} \quad \Delta = \begin{pmatrix} x & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{x} \end{pmatrix}$$

so that $g^* F g = F$ implies that $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^* F_0 \tilde{g} = F_0$.

This commensurability can be expressed more explicitly: Λ can be chosen so that $\tilde{\Pi}$ is in the normalizer Γ of Λ in $SU(2,1)$. Both $\tilde{\Pi}$ and Λ are of finite index in Γ.

It is shown in [PY] that $[\Gamma : \Lambda] = 3$.

Using $\mu(PU(2,1)/\Pi) = 1$, we find that

$$\mu(SU(2,1)/\Lambda) = \frac{1}{[\varphi(\Gamma) : \Pi]}.$$

But there is also Prasad's Covolume Formula, which in this context tells us that

$$\mu(SU(2,1)/\Lambda) = \frac{1}{D} \prod_{v \in \mathcal{T}} e'(P_v),$$

where $\mathcal{T} \subset V_f$ is finite, the $e'(P_v)$'s are certain positive integers depending on the order q_v of the residue field of k_v, and where D is as follows:

	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{C}_8$	$\mathcal{C}_{11}$	$\mathcal{C}_{18}$	$\mathcal{C}_{21}$
D	600	32	128	864	48	12

Comparing the two formulas, we get

$$D = [\varphi(\Gamma) : \Pi] \prod_{v \in \mathcal{T}} e'(P_v).$$

Any prime dividing D is either 2, 3 or 5. This severely restricts the possibilities for the P_v's. For example, under some conditions, $e'(P_v) = q_v^2 - q_v + 1$. It is elementary that unless $q_v = 2$, $q_v^2 - q_v + 1$ is divisible by a prime $p > 5$.

We find that the parahorics P_v must be maximal, or can be chosen to be maximal, for all v's.

When v splits in ℓ, any two maximal parahorics are conjugate by an element of $\overline{G}(k_v)$.

When v does not split in ℓ, there are two conjugacy classes of maximal parahorics, "type 1" and "type 2".

We find that at most one P_v can be of type 2.

When all such P_v's are of type 1, we can assume that $\varphi(\Gamma)$ is the following explicit group

$$\bar{\Gamma} = \{g \in M_{3\times 3}(\mathfrak{o}_\ell) : g^* F g = F\}/\{tI : t \in \mathfrak{o}_\ell \text{ and } |t| = 1\},$$

where F is defined above, and $\mathfrak{o}_\ell$ is the ring of algebraic integers in ℓ.

For each $\mathcal{C}_j$ there is another possibility for the group $\varphi(\Gamma)$, corresponding to a type 2 maximal parahoric group P_v for a particular v. For $\mathcal{C}_{21}$ there are two other possibilities, corresponding to a type 2 maximal parahoric group P_v for one or other of the two 2-adic places of $k = \mathbb{Q}(\sqrt{33})$.

A fundamental tool for [PY] is a result of Chern (called the Hirzebruch Proportionality Principle), valid for any torsion-free cocompact $\Pi \subset PU(2,1)$ and for $X = B(\mathbb{C}^2)/\Pi$, telling us that

$$\chi(X) = 3\text{vol}(\mathcal{F}_\Pi),$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X, where $\mathcal{F}_\Pi \subset B(\mathbb{C}^2)$ is a fundamental domain for the action of Π on $B(\mathbb{C}^2)$, and where vol is a suitably normalized volume on $B(\mathbb{C}^2)$, invariant under the action of $PU(2,1)$.

Since $\chi(X)$ is the alternating sum of the Betti numbers of X, for an fpp we have $\chi(X) = 3$, and so $\text{vol}(\mathcal{F}_\Pi) = 1$.

Let Π be a subgroup of index 864 in $\bar{\Gamma}$ in the $(\mathcal{C}_{11}, \emptyset)$ case. How do we check that Π is torsion-free?

In this case, we know that $d_2 = 2r_2$. Let $S = S_2 = \{g \in \bar{\Gamma} : d(0, g.0) \leq d_2\}$, Then

$$\mathcal{F}_{\bar{\Gamma}} = \mathcal{F}_2 = \{z \in B(\mathbb{C}^2) : d(0, z) \leq d(g.0, z) \text{ for all } g \in S\}.$$

Any $g \in \bar{\Gamma}$ of finite order must fix a point x of $B(\mathbb{C}^2)$. Conjugating g, we may assume $x \in \mathcal{F}_{\bar{\Gamma}}$. Then $d(0, g.0) \leq 2d(0, x) \leq 2r_2 = d_2$, so that $g \in S_2 = S$. Such g's lie in just 3 double cosets K, KbK and $Kbv^{-1}bK$.

We get a short list $g_1, \ldots, g_m$ of conjugacy class representatives of elements of finite order. Next we pick a transversal $t_1, \ldots, t_{864}$ for Π, e.g., $K \cup Kb \cup Kb^2$.

We need only check that $t_i g_j t_i^{-1} \notin \Pi$ for $i = 1, \ldots, 864$, and $j = 1, \ldots, n$.