Curvature of Graphs

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Motivations

1. Explore concepts of metric geometry in the context of graph theory

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2. Develop efficient tools for the qualitative analysis of empirical networks (from neurobiology, molecular biology, social systems,...)

Curvature in Riemannian geometry

Three types:[1]

- Scalar curvature
- Ricci curvature
- Sectional curvature

[1] J.J., Riemannian Geometry and Geometric Analysis, Springer, 6th ed., 2011

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- Scalar curvature $\longrightarrow$ assigned to points
- Ricci curvature $\longrightarrow$ assigned to tangent vectors
- Sectional curvature $\longrightarrow$ assigned to tangent planes

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Curvature of graphs

- Scalar curvature $\longrightarrow$ assigned to points
  $\longrightarrow$ assigned to vertices
- Ricci curvature $\longrightarrow$ assigned to tangent vectors
  $\longrightarrow$ assigned to directions=edges (2 vertices)
- Sectional curvature $\longrightarrow$ assigned to tangent planes
  $\longrightarrow$ assigned to triangles (3 vertices)

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Sectional curvature controls distances in triangles from above.

$\longrightarrow$ Upper bounds are geometrically powerful.

Manifolds of negative or nonpositive sectional curvature are geometrically well understood, whereas the geometry of those of positive or nonnegative curvature is still not clear.

Curvature in Riemannian geometry, ctd.

Example: Relation between volume of a ball and area of its boundary sphere

Ricci

With a lower Ricci curvature bound, the interior of a ball controls its boundary
$\longrightarrow$ from local to global

Sectional

With an upper sectional bound, the boundary of a ball controls its interior.
$\longrightarrow$ from asymptotic to local

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Such properties may hold in spite of local fluctuations.
Example: Gromov hyperbolicity in geometric group theory.

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Conclusion: Ricci and sectional curvature bounds play opposite roles in geometry.

Volume growth

$\mathrm{Ric} \geq 0$ implies (at most) polynomial volume growth (R.Bishop) and polynomial growth of finitely generated subgroups of π_1 (J.Milnor) (same growth rates by an earlier result of A.S.Schwarz), whereas $\mathrm{Sec} < 0$ implies exponential volume growth on universal cover (P.Günther) and of π_1 (J.Milnor).

Volume growth

$\mathrm{Ric} \geq 0$ implies that bounded harmonic functions are constant (S.T.Yau) and a dimension estimate for polynomial growth harmonic functions (Colding-Minicozzi, P.Li).

[2] B.B.Hua, J.J., Math.Z. 280:551–567 (2015); related work with B.B.Hua, S.P.Liu, X.Li-Jost

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Analogous results for Cayley graphs of finitely generated groups of polynomial growth. Such groups are virtually nilpotent (M.Gromov). Polynomial growth harmonic function theorem on Cayley graphs of such groups gives a new proof (B.Kleiner; quantitative version by Shalom-Tao).

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Theorem

Let (G, S) be a Cayley graph of a group of polynomial growth with the homogeneous dimension D. Then for $d \geq 1$, the space $H^d(G, S)$ of harmonic functions of degree d satisfies

$$\dim H^d(G, S) \leq C(S) d^{D-1}.$$

Holds also on graphs (with bounded geometry) roughly isometric to Cayley graphs of groups of polynomial growth.

[2] B.B.Hua, J.J., Math.Z. 280:551–567 (2015); related work with B.B.Hua, S.P.Liu, X.Li-Jost

Curvature of graphs: Forman-Ricci

Weighted graph with edge weights w_e and vertex weights w_v.

$v \sim w$: vertices v and w connected by an edge,

$e \sim f$: edges e and f share a vertex.

Forman's curvature [3] for an edge e connecting vertices v_1, v_2.

$$\mathrm{Ric}(e) = w_e \left(\frac{w_{v_1}}{w_e} + \frac{w_{v_2}}{w_e} - \sum_{e_{v_1} \sim e, e_{v_2} \sim e} \left[\frac{w_{v_1}}{\sqrt{w_e w_{e_{v_1}}}} + \frac{w_{v_2}}{\sqrt{w_e w_{e_{v_2}}}} \right] \right) \tag{1}$$

where e_{v_1}, e_{v_2} denote the set of edges $\neq e$ connected to vertices v_1 and v_2, resp.

[3] R.Forman, Discrete Comput. Geom. 29:323–374 (2003)

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where e_{v_1}, e_{v_2} denote the set of edges $\neq e$ connected to vertices v_1 and v_2, resp.
For an unweighted graph, simply

$$\mathrm{Ric}(e) = 2 - \mathrm{deg} v_1 - \mathrm{deg} v_2. \tag{2}$$

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For an unweighted graph, simply

$$\mathrm{Ric}(e) = 2 - \deg v_1 - \deg v_2. \tag{2}$$

Comes from a combinatorial Bochner formula. A graph with $\mathrm{Ric} > 0$ has $b_1 = 0$. Less trivial for higher dimensional simplicial complexes.

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Curvature of graphs: Forman-Ricci

Let now Γ be a *directed* graph, e an edge with tail v. We [4] define the Ricci curvature of e as

$$\mathrm{Ric}(e) = w_e \left(\frac{w_v}{w_e} - \sum_{e_v \sim e} \frac{w_v}{\sqrt{w_e w_{e_v}}} \right) \qquad (3)$$

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Together with the students Melanie Weber, R.P. Sreejith, Karthikeyan Mohanraj, we currently investigate the Ricci curvature properties of undirected and directed empirical networks. It turns out that Ricci curvature seems to be a good indicator of other, more global and hence more difficult to compute, properties of real networks.

[4] A.Samal, E.Saucan, J.J., to appear

Curvature of graphs: Ollivier-Ricci

$$\text{Degree } d_v := \#(\text{neighbors of } v)$$

$$m_v(v') := \begin{cases} \frac{1}{d_v} \text{ if } v' \sim v \\ 0 \text{ else.} \end{cases}$$

Wasserstein distance of measures m_v, m_w for $v \sim w$

$$W_1(m_v, m_w) := \min_{\xi \in \Pi(m_v, m_w)} \sum_{V \times V} \mathrm{dist}(v', w') \xi(v', w'),$$

where $\Pi(m_v, m_w)$ is the set of all measures with marginals μ and ν (transportations from m_v to m_w.
Optimal transport of neighborhood of v to that of w.

[5] Y. Ollivier, J. Funct.Anal. 256 (2009) 810-864

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Ollivier-Ricci curvature[5]

$$\kappa(v, w) := 1 - \frac{W_1(m_v, m_w)}{\mathrm{dist}(v, w)}. \quad (4)$$

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Curvature of graphs: Ollivier-Ricci

6

$$\begin{aligned}\#(v,w) &:= \#(\text{triangles with vertices } v,w)\\ &= \#(\text{common neighbors of } v,w).\end{aligned}$$

Theorem

$$\begin{aligned}\kappa(v,w) \geq \quad & -(1-\tfrac{1}{d_v}-\tfrac{1}{d_w}-\tfrac{\#(v,w)}{\min(d_v,d_w)})_+\\ & -(1-\tfrac{1}{d_v}-\tfrac{1}{d_w}-\tfrac{\#(v,w)}{\max(d_v,d_w)})_+\\ & +\tfrac{\#(v,w)}{\max(d_v,d_w)}\end{aligned}$$

and also

$$\kappa(v,w) \leq \frac{\#(v,w)}{\max(d_v,d_w)}.$$

Curvature of graphs: Ollivier-Ricci [6]

Theorem

$$\kappa(v,w) \geq -\left(1 - \frac{1}{d_v} - \frac{1}{d_w} - \frac{\#(v,w)}{\min(d_v,d_w)}\right)_+ -\left(1 - \frac{1}{d_v} - \frac{1}{d_w} - \frac{\#(v,w)}{\max(d_v,d_w)}\right)_+ + \frac{\#(v,w)}{\max(d_v,d_w)}$$

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Example: K_n: $\#(v,w) = n-2$, $\kappa(v,w) = \frac{n-2}{n-1}$.

[6] S.P.Liu, J.J., Discrete Comput.Geom. 51, 300–322 (2014)

Curvature of graphs: Ollivier-Ricci [6]

Theorem

$$\kappa(v,w) \geq -\left(1 - \frac{1}{d_v} - \frac{1}{d_w} - \frac{\#(v,w)}{\min(d_v,d_w)}\right)_+ - \left(1 - \frac{1}{d_v} - \frac{1}{d_w} - \frac{\#(v,w)}{\max(d_v,d_w)}\right)_+ + \frac{\#(v,w)}{\max(d_v,d_w)}$$

and also

$$\kappa(v,w) \leq \frac{\#(v,w)}{\max(d_v,d_w)}.$$

Triangles, quadrangles and pentagons containing v and w help to reduce the transportation cost. Without such short cycles, neighbors of v and w (other than w and v themselves) have distance $= 3$.

[6] S.P.Liu, J.J., Discrete Comput.Geom. 51, 300–322 (2014)

Curvature of graphs: Ollivier-Ricci

$$\kappa(v,w) \quad = \quad 1 - \frac{W_1(m_v, m_w)}{\text{dist}(v,w)} \text{ assume } > k.$$

Eigenvalues of Δ satisfy (Ollivier)

$$k \leq \lambda \leq 2 - k.$$

[7] F.Bauer, J.J., S.P.Liu, Math.Res.Lett.19, 1185–1205 (2012)

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Unfortunately, $k \le 0$ for most graphs.

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Improve estimate[7] by considering neighborhood graph of order t, with weights $w_{v,w}$ given by probabilies for reaching w from v in t steps,

$$1-(1-k[t])^{1/t} \leq \lambda \leq 1+(1-k[t])^{1/t},$$

where

$$k[t] > 0 \text{ for } t \gg 0 \qquad \text{unless } \Gamma \text{ is bipartite.}$$

[7] F.Bauer, J.J., S.P.Liu, Math.Res.Lett.19, 1185–1205 (2012)

Sectional curvature

3 aspects of *nonpositive* curvature

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3. **3 can easily meet:** The smallest maximal distance of a point to three given points is not larger than it would be in a Euclidean comparison triangle.

(For curvature $\leq K$, instead of ≤ 0, we use a surface of constant curvature K in place of the Euclidean plane.)

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Let u, v, w be vertices. Γ has nonpositive sectional curvature (in the sense of Alexandrov) if for all $u' \in \gamma(v,w)$,

$$\begin{aligned} \mathrm{dist}(u,u') \leq \quad & (1 - \tfrac{\mathrm{dist}(v,u')}{\mathrm{dist}(v,w)})\mathrm{dist}^2(v,u) \\ & + \tfrac{\mathrm{dist}(v,u')}{\mathrm{dist}(v,w)}\mathrm{dist}^2(w,u) \\ & - \tfrac{\mathrm{dist}(v,u')}{\mathrm{dist}(v,w)}(1 - \tfrac{\mathrm{dist}(v,u')}{\mathrm{dist}(v,w)})\mathrm{dist}^2(v,w), \end{aligned}$$

that is, if the shortest path from v to w is not farther away from another vertex u than were the case in a Euclidean triangle with the same side lengths.

Sectional curvature of graphs

Alternative definition[8]:

$$\min_{s \in V} \max(\mathrm{dist}(s,u), \mathrm{dist}(s,v), \mathrm{dist}(s,w))$$

is not larger than the corresponding quantity in Euclidean space.

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This definition also works on disconnected metric spaces.
In order to capture asymptotic aspects, one may allow for an error ε that is independent of the distances between the vertices u, v, w.

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