Buser inequalities for graph connection Laplacian

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joint work with

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"Geometry and Computation on Groups and Complexes"

Durham University

Plan

- Prologue
- Connection Laplacian
- Cheeger type constants
- A problem from spectral graph theory
- Buser inequalities
- Understandings of the curvature condition

Prologue

Basic settings

- $G = (V, E)$: an undirected finite graph.
- The set E^{or} of all oriented edges:

$$E^{or} := \{(x, y), (y, x) | \{x, y\} \in E\}.$$

- H: a group.

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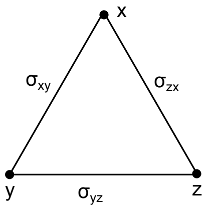
Definition

A *signature* of G is a map $\sigma : E^{or} \to H$ such that

$$\sigma((x, y)) = \sigma((y, x))^{-1}.$$

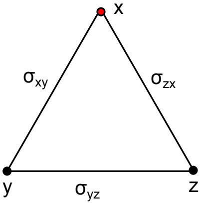
We will write $\sigma_{xy} := \sigma((x, y))$ for short.



Balanced signatures

Consider a walker:

$$w(x) = id \curvearrowright w(y) = \sigma_{xy} \curvearrowright w(z) = \sigma_{xy}\sigma_{yz}$$
$$\curvearrowright w(x) = \sigma_{xy}\sigma_{yz}\sigma_{zx} \underbrace{=id}_{?}$$



Definition

We say σ is a *balanced* signature of G if for any cycle $(x_1, x_2), (x_2, x_3), \ldots, (x_\ell, x_1)$, we have

$$\sigma_{x_1x_2}\sigma_{x_2x_3}\cdots\sigma_{x_\ell x_1} = id \in H.$$

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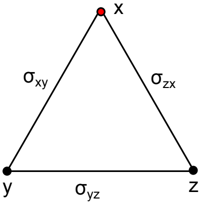

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- $\sigma_{\mathrm{triv}} : E^{or} \to id \in H$ is balanced.
- When $H = O(1) = \{\pm 1\}$, Harary's signed graph ('53).

An equivalent definition of balancedness

Definition

Given a function $\tau : V \to H$ and a signature σ, we consider the new signature σ^τ defined by

$$\sigma^\tau_{xy} := \tau(x)^{-1}\sigma_{xy}\tau(y), \ \forall\, (x, y) \in E^{or}.$$

We call the function τ a *switching function*. Two signatures σ and σ' are said to be *switching equivalent*, if there exists a switching function τ such that $\sigma' = \sigma^\tau$.

Proposition

σ is balanced $\Leftrightarrow \exists\ \tau$ such that $\sigma^\tau = \sigma_{\mathrm{triv}} : E^{or} \to id \in H$.

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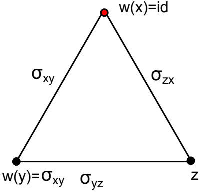
$$\sigma_{xy}\sigma_{yz}\sigma_{zx} = id \ \Leftrightarrow\ \sigma^{w^{-1}} = \sigma_{\mathrm{triv}}.$$



A third way to detect balancedness

$$\sigma \text{ is balanced } \Leftrightarrow \exists\, \tau \text{ such that } \tau(x)^{-1}\sigma_{xy}\tau(y) = id,\ \forall\,(x,y) \in E^{or}$$
$$\Leftrightarrow \exists\, \tau \text{ such that } \tau(x) = \sigma_{xy}\tau(y),\ \forall\,(x,y) \in E^{or}$$

Definition (Connection Laplacian Δ^σ)

For any $\tau : V \to H$, and $x \in V$, we have

$$\Delta^\sigma \tau(x) := \frac{1}{d_x} \sum_{y : y \sim x} (\tau(x) - \sigma_{xy}\tau(y)).$$

(We have to make sense to sum up two group elements.)

$$\sigma \text{ is balanced } \Rightarrow \exists\, \tau \text{ such that } \Delta^\sigma \tau = 0.$$

This is the main topic of this talk.

Connection Laplacian

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- $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
- vector-valued functions $f : V \to \mathbb{K}^d$.

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$$\Delta^\sigma f(x) := \frac{1}{d_x} \sum_{y:y\sim x} (f(x) - \sigma_{xy} f(y)).$$

As a matrix,

$$\Delta^\sigma = I_{Nd} - D^{-1}A^\sigma = I_{Nd} - \begin{bmatrix} d_{x_1} I_d & & & \\ & d_{x_2} I_d & & \\ & & \ddots & \\ & & & d_{x_N} I_d \end{bmatrix}^{-1} \begin{bmatrix} 0_d & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_N} \\ \sigma_{x_2x_1} & 0_d & \cdots & \sigma_{x_2x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_Nx_1} & \sigma_{x_Nx_2} & \cdots & 0_d \end{bmatrix}.$$

Δ^σ is Hermitian with eigenvalues

$$0 \le \underbrace{\lambda_1^\sigma \le \lambda_2^\sigma \le \cdots \le \lambda_d^\sigma}_{1} \le \cdots \le \underbrace{\lambda_{(N-1)d+1}^\sigma \le \lambda_{(N-1)d+2}^\sigma \le \cdots \le \lambda_{Nd}^\sigma}_{N} \le 2.$$

Switching invariants

Let $\sigma^{\tau}_{xy} := \tau(x)^{-1}\sigma_{xy}\tau(y)$, We have

$$\Delta^{\sigma^{\tau}} = \begin{bmatrix} \tau(x_1) & & & \\ & \tau(x_2) & & \\ & & \ddots & \\ & & & \tau(x_N) \end{bmatrix}^{-1} \cdot \Delta^{\sigma} \cdot \begin{bmatrix} \tau(x_1) & & & \\ & \tau(x_2) & & \\ & & \ddots & \\ & & & \tau(x_N) \end{bmatrix}$$

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Eigenvalues of Δ^{σ} are switching invariant.

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Proposition

σ *is balanced* $\Leftrightarrow$ $\lambda_1^{\sigma} = \lambda_2^{\sigma} = \cdots = \lambda_d^{\sigma} = 0.$

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Proposition

σ is balanced $\Leftrightarrow$ $\lambda_1^{\sigma} = \lambda_2^{\sigma} = \cdots = \lambda_d^{\sigma} = 0$.

Proof.

$\Rightarrow$: Δ^{triv} has 0 eigenvalues corr. to constant vector-valued functions.
$\Leftarrow$: Eigenfcts of 0 eigenvalues of Δ^{σ} give the switching function. $\square$

Frustration index and Cheeger type constants

Measuring how far σ is from being balanced

Cheeger type constants

Definition (Frustration index)

We define the frustration index $\iota^\sigma(G)$ of (G,σ) as

$$\begin{aligned}\iota^\sigma(G) :&= \min_{\tau:V\to H} \sum_{\{x,y\}\in E} \|\tau(x)-\sigma_{xy}\tau(y)\| \\ &= \min_{\tau:V\to H} \sum_{\{x,y\}\in E} \|id-\sigma_{xy}^\tau\|.\end{aligned}$$

We have freedom to choose a matrix norm $\|\cdot\|$. We used $\|\tau(x)\|=1$.

$$\sigma \text{ is balanced} \quad \Leftrightarrow \quad \iota^\sigma(G)=0.$$

Cheeger type constants

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$$\sigma \text{ is balanced} \quad \Leftrightarrow \quad \iota^\sigma(G) = 0.$$

Definition (Cheeger type constants)

We define

$$h_1^\sigma := \min_{\{S_i\}_{i=1}^k} \max_{1 \le i \le k} \frac{\iota^\sigma(S_i) + |E(S_i, V \setminus S_i)|}{\sum_{x \in S_i} d_x},$$

$\{S_i\}_{i=1}^k$: nonempty, pairwise disjoint subsets of V.

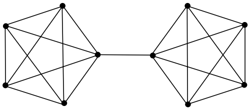
Digest of the Cheeger type constants

Consider special cases: $H = O(1) = \{\pm 1\}$.

- $\sigma = \sigma_{\mathrm{triv}} : E^{or} \to +1 \in H$.

$$h_1 = 0. \text{ (Set } S = V.\text{)}$$

$$h_2 = \min_{S_1 \sqcup S_2 = V} \max_{i=1,2} \frac{|E(S_i, V \setminus S_i)|}{\sum_{x \in S_i} d_x}.$$



h_2 is the classical Cheeger constant.

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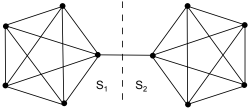

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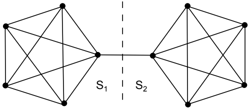


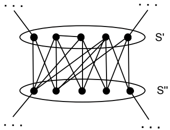
h_2 is the classical Cheeger constant.

- $\sigma : E^{or} \to -1 \in H$

$$\iota^\sigma(S) = \min_{\tau : S \to \{\pm 1\}} \sum_{\{x,y\} \in E_S} |\tau(x) - \underbrace{\sigma_{xy}}_{=-1} \tau(y)|$$

$$= \min_{S' \sqcup S'' = S} (2|E_{S'}| + 2|E_{S''}|).$$

$$\widehat{h} := h_1^\sigma = \min_{S' \sqcup S'' = S \subseteq V} \frac{2|E_{S'}| + 2|E_{S''}| + |E(S, V \setminus S)|}{\sum_{x \in S} d_x}.$$



$\widehat{h}$ is the dual Cheeger constant/ bipartiteness ratio

(Bauer-Jost, Trevisan)

Problem

$$0 \le \underbrace{\lambda_1^\sigma \le \lambda_2^\sigma \le \cdots \le \lambda_d^\sigma}_{h_1^\sigma\,?} \le \cdots \le \underbrace{\lambda_{(N-1)d+1}^\sigma \le \lambda_{(N-1)d+2}^\sigma \le \cdots \le \lambda_{Nd}^\sigma} \le 2.$$

A problem from spectral graph theory

Classical Laplacian and its eigenvalues

Consider special cases: $\sigma : E^{or} \to +1 \in O(1)$. $f : V \to \mathbb{R}$.

$$\Delta f(x) = \frac{1}{d_x} \sum_{y:y\sim x} (f(x) - f(y)).$$

$$0 = \underline{\lambda_1 \leq \lambda_2} \leq \cdots \leq \underline{\lambda_N \leq 2}.$$

- $\lambda_2 = 0$ iff G is disconnected.

$$\frac{h_2^2}{2} \leq \lambda_2 \leq 2h_2 \quad \text{(Cheeger inequlity, Alon-Milman, Dodziuk...)}$$

- $\lambda_N = 0$ iff G has a bipartite connected component.

$$\frac{\widehat{h}^2}{2} \leq 2 - \lambda_N \leq 2\widehat{h} \quad \text{(dual Cheeger inequlity, Bauer-Jost, Trevisan)}$$

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Problem: (i) When will $\lambda_2 \approx h_2^2$? (ii) When will $2 - \lambda_N \approx \widehat{h}^2$?

Buser inequality

Definition (Bakry-Émery)

For any two functions $f, g : V \to \mathbb{R}$, we define two operators Γ and Γ_2:

$$2\Gamma(f,g) := -\left[\Delta(fg) - f\Delta g - (\Delta f)g\right],$$
$$2\Gamma_2(f,g) := -\left[\Delta(\Gamma(f,g)) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\right].$$

The graph G *satisfies* $CD(K,\infty)$ if we have for any function $f : V \to \mathbb{R}$,

$$\Gamma_2(f,f) \geq K\Gamma(f,f).$$

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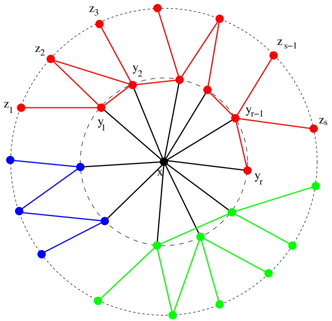
Theorem (Klartag-Kozma-Ralli-Tetali,'16)

If G satisfies the curvature dimension inequality $CD(0,\infty)$, then

$$\lambda_2 \leq 16 d_{\max} h_2^2. \quad \textit{(Buser type inequality)}$$

Digest of the curvature condition

- All abelian Cayley graphs satisfy $CD(0,\infty)$.(Chung-Yau '96, Lin-Yau '10, Klartag-Kozma-Ralli-Tetali '16)
- If both G_1 and G_2 satisfy $CD(0,\infty)$, so does their Cartesian product $G_1 \times G_2$. (L.-Peyerimhoff '14+)
- Curvature and local connectedness. (Assume G be regular.) (Cushing-L.-Peyerimhoff '16+)



Dual Buser inequality?

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Note that

$$2f(x) - \Delta f(x) = \frac{1}{d_x} \sum_{y: y \sim x} (f(x) + f(y)) = \Delta^{\sigma} f(x),$$

where $\sigma : E^{or} \to -1 \in O(1)$, and

$$2 - \lambda_N(\Delta) = \lambda_1^{\sigma}(\Delta^{\sigma})!$$

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Buser type inequality for connection Laplacian?

Buser inequalities for connection Laplacian

Curvature condition for connection Laplacian

Curvature condition

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Curvature condition

Definition (L.-Münch-Peyerimhoff)

For any two functions $f, g : V \to \mathbb{K}^d$, we define two operators Γ^σ and Γ_2^σ:

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The graph G with σ *satisfies* $CD^\sigma(K,\infty)$ if we have for any function $f : V \to \mathbb{K}^d$,

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$$2\Gamma^\sigma(f,g)(x) = \frac{1}{d_x}\sum_{y,y\sim x}(\sigma_{xy}f(y) - f(x))^\top(\overline{\sigma_{xy}g(y) - g(x)});$$

Buser inequality

$$0 \leq \underbrace{\lambda_1^\sigma \leq \lambda_2^\sigma \leq \cdots \leq \lambda_d^\sigma}_{h_1^\sigma ?} \leq \cdots \leq \underbrace{\lambda_{(N-1)d+1}^\sigma \leq \lambda_{(N-1)d+2}^\sigma \leq \cdots \leq \lambda_{Nd}^\sigma} \leq 2.$$

Theorem (L.-Münch-Peyerimhoff '15+)

Assume that a graph G with a signature σ satisfies $CD^\sigma(0,\infty)$. Then for each natural number $1 \leq k \leq N$, we have

$$\lambda_{kd}^\sigma \leq 16 d_{\max} (kd)^2 \log(2kd) \cdot (h_k^\sigma)^2.$$

Buser inequality

$$0 \le \underbrace{\lambda_1^\sigma \le \lambda_2^\sigma \le \cdots \le \lambda_d^\sigma}_{h_1^\sigma ?} \le \cdots \le \underbrace{\lambda_{(N-1)d+1}^\sigma \le \lambda_{(N-1)d+2}^\sigma \le \cdots \le \lambda_{Nd}^\sigma} \le 2.$$

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Assume that a graph G with a signature σ satisfies $CD^\sigma(0,\infty)$. Then for each natural number $1 \le k \le N$, we have

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Assume that G satisfies $CD^{-\sigma_{\mathrm{triv}}}(0,\infty)$. Then we have

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Recall: G satisfies $CD(0,\infty) \Rightarrow \lambda_2 \leq 16 d_{\max} h_2^2$. Klartag-Kozma-Ralli-Tetali

Understandings of the curvature condition

Curvature condition (I): Switching invariant

Proposition

If (G, σ) satisfies $CD^{\sigma}(K, \infty)$, then (G, σ^{τ}) satisfies $CD^{\sigma^{\tau}}(K, \infty)$ for any switching function $\tau : V \to H$.

Proof.

We have for any $\tau : V \to H$ and $f, g : V \to \mathbb{K}^d$,

$$\Gamma^{\sigma^{\tau}}(f, g) = \Gamma^{\sigma}(\tau^{-1}f, \tau^{-1}g) \text{ and } \Gamma_2^{\sigma^{\tau}}(f, g) = \Gamma_2^{\sigma}(\tau^{-1}f, \tau^{-1}g),$$

Recall $CD^{\sigma}(K, \infty) \Leftrightarrow \Gamma_2^{\sigma}(f, f) \geq K\Gamma^{\sigma}(f, f)\ \forall f.$ $\square$

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Proposition

Let (G, σ) be given. If every cycle of length 3 or 4 is balanced (as a subgraph), then

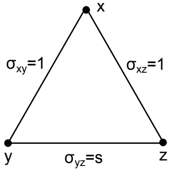
$$(G, \sigma) \text{ satisfies } CD^{\sigma}(K, \infty) \Leftrightarrow G \text{ satisfies } CD(K, \infty).$$

Curvature condition (II): Short cycle matters

Consider special case: $H = U(1) = \{z \in \mathbb{C}, |z| = 1\}$.

Any $\sigma : E^{or}_{\mathcal{C}_3} \to U(1)$ is switching equivalent to $\to$

Let $K_\infty(s)$ be the largest one such that $CD^\sigma(K, \infty)$ holds.



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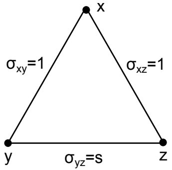
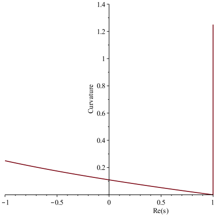


$$K_\infty(s) = \begin{cases} \frac{5}{4}, & \text{if } s = 1; \\ \frac{5 - \sqrt{17 + 8\mathrm{Re}(s)}}{8}, & \text{otherwise.} \end{cases}$$



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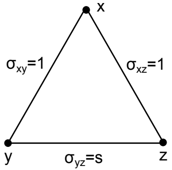

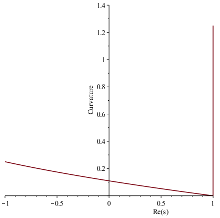

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Moreover,

$$K_\infty(\mathcal{C}_4, \sigma) = \begin{cases} 1, & \text{if } s = 1; \\ 0, & \text{otherwise,} \end{cases}$$

and, for $N \geq 5$,

$$K_\infty(\mathcal{C}_N, \sigma) = 0.$$

Curvature condition (III): Cartesian Product

Two graphs $G_i = (V_i, E_i)$ with signatures $\sigma_i : E_i^{or} \to H_i = U(d_i)$, $i = 1, 2$.

Cartesian product $G_1 \times G_2 = (V_1 \times V_2, E_{12})$.

Assign a signature $\widehat{\sigma}_{12} : E_{12}^{or} \to H_1 \otimes H_2$ to $G_1 \times G_2$:

$$\widehat{\sigma}_{12,(x_1,y)(x_2,y)} := \sigma_{1,x_1x_2} \otimes \mathrm{I}_{d_2}, \quad \text{for any} \quad (x_1, x_2) \in E_1^{or}, y \in V_2;$$
$$\widehat{\sigma}_{12,(x,y_1)(x,y_2)} := \mathrm{I}_{d_1} \otimes \sigma_{2,y_1y_2}, \quad \text{for any} \quad (y_1, y_2) \in E_2^{or}, x \in V_1.$$

Theorem (L.-Münch-Peyerimhoff)

Let G_i with $\sigma_i, i = 1, 2$ satisfy $CD^{\sigma_1}(0, \infty)$ and $CD^{\sigma_2}(0, \infty)$, respectively. Then the Cartesian product $G_1 \times G_2$ with $\widehat{\sigma}_{12}$ satisfies $CD^{\widehat{\sigma_{12}}}(0, \infty)$.

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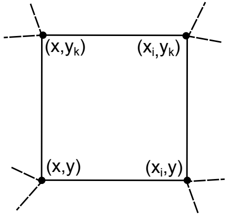

$$(\sigma_{1,xx_i} \otimes \mathrm{I}_{d_2}) \cdot (\mathrm{I}_{d_1} \otimes \sigma_{2,yy_k}) \cdot (\sigma_{1,x_ix} \otimes \mathrm{I}_{d_2}) \cdot (\mathrm{I}_{d_1} \otimes \sigma_{2,y_ky}) = id$$

Curvature condition (IV): Heat equation

We derive characterizations of the CD^{σ} inequality via the solution of the following associated continuous time heat equation,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\Delta^{\sigma} u(x,t), \\ u(x,0) = f(x), \end{cases}$$

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The solution $u : V \times [0,\infty) \to \mathbb{K}^d$ is given by

$$P_t^\sigma f := e^{-t\Delta^\sigma} f.$$

Clearly, we have $P_0^\sigma f = f$.

Curvature condition (IV): Heat equation

Key result for our proof of Buser inequalities:

Theorem

Let (G, σ) be given. TFAE:

(i) *The inequality $CD^\sigma(K, \infty)$ holds.*

(ii) *For any function $f : V \to \mathbb{K}^d$ and $t \geq 0$, we have*

$$\Gamma^\sigma(P_t^\sigma f) \leq e^{-2Kt} P_t(\Gamma^\sigma(f));$$

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Proof.

(i) $\Rightarrow$ (ii): For any $0 \leq s \leq t$, we consider

$$F(s) := e^{-2Ks} P_s(\Gamma^\sigma(P_{t-s}^\sigma f)).$$

Note $F(0) = \Gamma^\sigma(P_t^\sigma f)$ and $F(t) = e^{-2Kt} P_t(\Gamma^\sigma(f))$.

$$\frac{d}{ds} F(s) = 2e^{-2Ks} \underbrace{P_s}_{\textit{nonnegative}} \left[\Gamma_2^\sigma(P_{t-s}^\sigma f) - K\Gamma^\sigma(P_{t-s}^\sigma f)\right] \geq 0,$$

□

"Jump" of the curvature

Lichnerowicz type estimate

Suppose G is connected. Let λ^σ be a nonzero eigenvalue of the connection Laplacian Δ^σ.

Theorem (L.-Münch-Peyerimhoff)

Assume G with a signature σ satisfies $CD^\sigma(K, \infty)$. Then

$$\lambda^\sigma \geq K.$$

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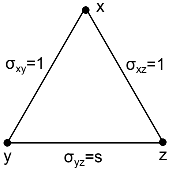

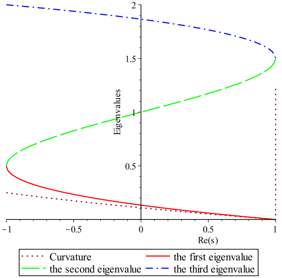

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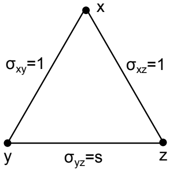

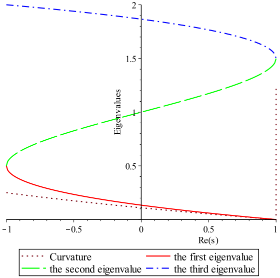

If $K_\infty(G, \sigma_{\mathrm{triv}}) > 0$, then the curvature, as a function of σ, will jump.

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Thank you for your attentions!