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Geometry III/IV

Problems, set 1.

Exercise 1. A norm $\|\cdot\| : V \to [0,\infty)$ on an $\mathbb{R}$-vector space V satisfies the properties

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

Prove the following facts:

(a) If a norm is induced from an inner product via $\|x\| = \sqrt{\langle x, x\rangle}$, then it satisfies the parallelogram equation

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

In this case, express the inner product $\langle x, y\rangle$ of two vectors solely in terms of the norm.

(b) There are two functions $f, g \in C([0,1])$ such that the maximum norm $\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$ violates the parallelogram equation.

Exercise 2. Recall that the norm and the distance function are related to the standard inner product of $\mathbb{R}^n$ via

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad d(x,y) = \|x - y\|.$$

(a) Show the Cauchy-Schwarz inequality, i.e.,

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \quad \forall\, x, y \in \mathbb{R}^n.$$

(b) Show that the triangle inequality of the distance function is a consequence of the Cauchy-Schwarz inequality.

Exercise 3. Prove the following properties of reflections s_{l_1}, s_{l_2}:

(a) If l_1 and l_2 are parallel lines with $l_1 = l_2 + v$ and v orthogonal to l_1, l_2, then

$$s_{l_1} \circ s_{l_2} = t_{2v}.$$

(b) If l_1 and l_2 intersect at an angle $\alpha \in (0, \pi)$ in the point $z \in \mathbb{R}^2$, i.e., l_1 is the counter clockwise rotation of l_2 by the angle α, then

$$s_{l_1} \circ s_{l_2} = r_{2\alpha,z}.$$

(c) Conclude that every isometry of $\mathbb{R}^2$ is the composition of at most three reflections. Which isometries cannot be written by less than three reflections? Justify your answer.