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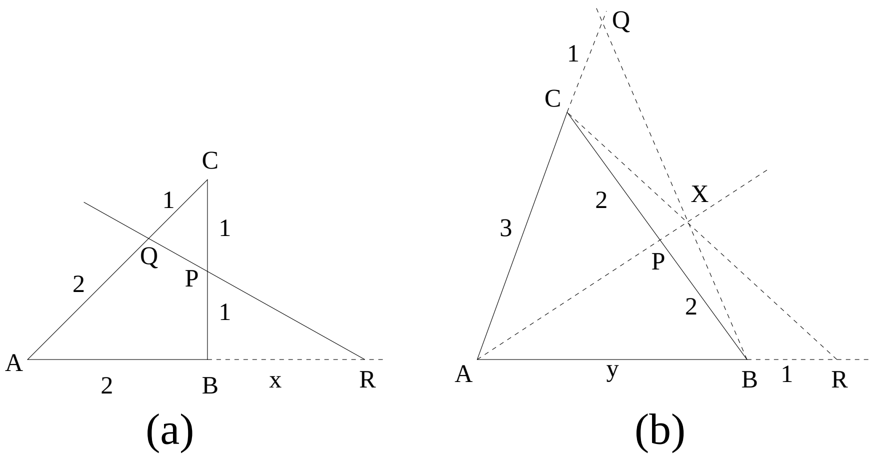
Geometry III/IV

Problems, set 6 (to be handed in on 1 December 2008).

Exercise 13. Calculate the unknown lengths x, y in the following pictures:



Exercise 14. Let

$$A = [1, 0, 2],\ B = [0, 1, 0],\ C = [1, -1, 0],\ D = [2, 1, 1]\ \in \mathbb{R}P^2.$$

Show that no three of these four points are collinear. Find $\alpha_j, \gamma_j \in \mathbb{R}$ such that

$$\begin{aligned} AB &= \{[x_1, x_2, x_3] \mid \sum_j \alpha_j x_j = 0 \,\text{and}\, (x_1, x_2, x_3) \neq 0\}, \\ CD &= \{[x_1, x_2, x_3] \mid \sum_j \gamma_j x_j = 0 \,\text{and}\, (x_1, x_2, x_3) \neq 0\}. \end{aligned}$$

Calculate the intersection point $AB \cap CD$.

Exercise 15. This exercise shows that $\mathbb{C}P^1$ coincides topologically with the two-sphere S^2. Let

$$\Phi : \mathbb{C}^2 \backslash \{0\} \to \mathbb{R}^3, \quad \Phi(z, w) \mapsto \left(\frac{2\mathrm{Re}(z\bar{w})}{|z|^2 + |w|^2}, \frac{2\mathrm{Im}(z\bar{w})}{|z|^2 + |w|^2}, \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2} \right).$$

(a) Show that $\|\Phi(z,w)\| = 1$ for all $(z,w) \in \mathbb{C}^2 \backslash \{0\}$.

(b) Show that $\Phi(\lambda z, \lambda w) = \Phi(z,w)$ for all $(z,w) \in \mathbb{C}^2 \backslash \{0\}$, $\lambda \in \mathbb{C} \backslash \{0\}$.

(a) and (b) imply that Φ induces a map

$$\Phi_0 : \mathbb{C}P^1 \to S^2, \quad \Phi_0([z,w]) \mapsto \left(\frac{2\mathrm{Re}(z\bar{w})}{|z|^2 + |w|^2}, \frac{2\mathrm{Im}(z\bar{w})}{|z|^2 + |w|^2}, \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2} \right).$$

(c) Show that Φ_0 is injective.

(d) Using $\Phi_0([1,0]) = (0,0,1)$ and

$$\Phi_0([z,1]) = \left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right),$$

show that Φ_0 is surjective.

The map Φ_0 is a bijective continuous map between $\mathbb{C}P^1$ and S^2.