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Geometry III/IV

Problems, set 7.

Exercise 16. Let $P_1, P_2, Q_1, Q_2, R_1, R_2 \in P^2(\mathbb{R})$ be points in the real projective plane with $P_1 \neq P_2$, $Q_1 \neq Q_2$ and $R_1 \neq R_2$. Let $P_j = [u_j]$, $Q_j = [v_j]$ and $R_j = [w_j]$. Show that P_1P_2, Q_1Q_2 and R_1R_2 are concurrent, i.e.,

$$P_1P_2 \cap Q_1Q_2 \cap R_1R_2 \neq \emptyset,$$

iff

$$\det \left(u_1 \times u_2 \quad v_1 \times v_2 \quad w_1 \times w_2 \right) = 0.$$

Here the matrix is formed by inserting the three cross product vectors as columns into the matrix.

Exercise 17.

(a) Find a projective transformation $f : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which maps the points

$$P_0 = [1, 2, 0],\ P_1 = [0, 1, 0],\ P_2 = [-1, 0, 2],\ P_3 = [-1, 3, 4]$$

to the points

$$Q_0 = [1, 0, 0],\ Q_1 = [1, 1, 0],\ Q_2 = [0, -1, 1],\ Q_3 = [1, 1, 1],$$

respectively.

(b) Find a projective transformation $g : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which maps the points P_0, P_1, P_2, P_3 from (a) to the points P_1, P_2, P_3, P_0, respectively (i.e. g interchanges these points).

(c) Calculate $g \circ f$.

Exercise 18. Let two triangles $\Delta P_0P_1P_2$ and $\Delta Q_0Q_1Q_2$ in the Euclidean plane $\mathbb{R}^2$ be given. Assume that all six points $P_0, \cdots, Q_2$ are different. Assume that the three lines P_0Q_0, P_1Q_1 and P_2Q_2 intersect in a common point $Z \in \mathbb{R}^2$. Now, any pair of lines P_iP_j, Q_iQ_j can either intersect or they can be parallel. Use Desargues Theorem of Projective Goemetry to describe all the configurations that can occur and to state the corresponding conclusions, i.e., formulate a Euclidean version of Desargues Theorem (which is a bit more complicated since you have to distiguish different situations).

Exercise 19. Let four projective points $P_1 = [v_1], P_2 = [v_2]$ and $Q_1 = [w_1], Q_2 = [w_2]$ in $P^2(\mathbb{R})$ be given and assume that $P_1 \neq P_2$, $Q_1 \neq Q_2$. Assume further that Q_1 does not lie in the projective line P_1P_2. Let $Z = P_1P_2 \cap Q_1Q_2$. Now, $Z = [u]$, and $u \in \mathbb{R}^3$ can be given by a formula which only uses v_1, v_2, w_1, w_2 and cross products. Derive this formula.