# Geometry III/IV: MATH 3201/4141

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17 December, 2008

## Contents



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# 1 Euclidean Geometry

## 1.1 General form of isometries of $\mathbb{R}^2$

Klein's idea: *Understand geometry by looking at the group of transformations preserving key properties of this geometry*

In *Euclidean Geometry*, we start with $\mathbb{R}^n$ and its *inner product*

$$\langle x, y\rangle = x^\top y = \sum_{i=1}^n x_i y_i,$$

where we consider $x, y$ as column vectors.

Properties of the inner product:

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0$ is equivalent to $x = 0$

The inner product induces a *norm* $\|x\| = \sqrt{\langle x, x \rangle}$ and a *distance function* $d(x, y) = \|x - y\| \geq 0$.

Properties of the distance function:

- $d(x, y) = d(y, x)$
- $d(x, y) \geq 0$
- $d(x, y) = 0$ is equivalent to $x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

**Definition 1.1.** *$f : \mathbb{R}^n \to \mathbb{R}^n$ is called an* isometry *if $f$ is surjective and if*

$$d(f(x), f(y)) = d(x, y).$$

*Natural Question:* What are the isometries of $\mathbb{R}^n$?

**Example.** *Let $A \in O(n) = \{C \in M(n, \mathbb{R}) \mid C^\top C = \mathrm{Id}\}$, $b \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax + b$. We show that $f$ is an isometry. $f$ is surjective: Let $y \in \mathbb{R}^n$ be given. We have to solve $f(x) = Ax + b = y$. The solution is $x = A^{-1}(y - b) = A^\top(y - b)$. It remains to show the following:*

$$\begin{aligned} d(f(x), f(y))^2 &= \|f(x) - f(y)\|^2 = \|(Ax + b) - (Ay + b)\|^2 = \|A(x - y)\|^2 \\ &= \langle A(x - y), A(x - y) \rangle = (x - y)^\top A^\top A(x - y) \\ &= (x - y)^\top (x - y) = \langle x - y, x - y \rangle = \|x - y\|^2 = d(x, y)^2. \end{aligned}$$

We will see later that these are all isometries of $\mathbb{R}^n$.

**Lemma 1.2.** *Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is injective.*

*Proof.* Assume $f(x) = f(y)$. Then

$$0 = d(f(x), f(y)) = d(x, y),$$

i.e., $x = y$. ∎

**Lemma 1.3.** *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, so is $f^{-1}$.*

*Proof.* Since $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ exists and is also bijective. Thus, $f^{-1}$ is surjective. To show:

$$d(f^{-1}(x), f^{-1}(y)) = d(x, y) \quad \forall x, y \in \mathbb{R}^n.$$

But

$$d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x)), f(f^{-1}(y))) = d(x, y),$$

since $f$ is an isometry. $\square$

**Lemma 1.4.** *If $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are isometries, so is $f \circ g : \mathbb{R}^n \to \mathbb{R}^n$.*

*Proof.* Since $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are bijective, $f \circ g : \mathbb{R}^n \to \mathbb{R}^n$ is also bijective, and thus surjective. To show:

$$d(f \circ g(x), f \circ g(y)) = d(x, y).$$

This follows immediately from the facts that $f, g$ are isometries:

$$d(x, y) = d(g(x), g(y)) = d(f(g(x)), f(g(y))) = d(f \circ g(x), f \circ g(y)).$$

$\square$

*Important consequence:* The set of all isometries of $\mathbb{R}^n$, denoted by $I(\mathbb{R}^n)$, forms a group. *Klein's viewpoint:* to understand Euclidean geometry means to understand the group $I(\mathbb{R}^n)$ of transformations preserving the distance $d$.

Our first goal is to prove the following:

**Theorem 1.5.** *Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form*

$$f(x) = Ax + b$$

*with $A \in O(n)$ and $b \in \mathbb{R}^n$.*

This is done in steps.

**Lemma 1.6.** *Assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry with $g(0) = 0$. Then $g$ is uniquely determined by its values of $g(e_1), g(e_2), \dots, g(e_n) \in \mathbb{R}^n$, where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.*

*Proof.* Let $g, h : \mathbb{R}^n \to \mathbb{R}^n$, $g(0) = h(0) = 0$ and $g(e_i) = h(e_i)$. We have to show that $g = h$. We consider the isometry $k : h^{-1} \circ g$. Then $k(0) = 0$ and $k(e_i) = e_i$, and it suffices to show that $k = \mathrm{id}$. Let $y = k(x)$, $x = (x_1, \dots, x_n)^\top$, $y = (y_1, \dots, y_n)^\top$.

a) We have $\|x\| = \|y\|$:

$$\|y\| = d(y, 0) = d(k(x), k(0)) = d(x, 0) = \|x\|.$$

b) We now show that $\|y - e_i\| = \|x - e_i\|$:

$$\|y - e_i\|^2 = d(y, e_i)^2 = d(k(x), k(e_i))^2 = d(x, e_i)^2 = \|x - e_i\|^2.$$

c) We have

$$\|y - e_i\|^2 = \langle y - e_i, y - e_i \rangle = \|y\|^2 - 2\langle y, e_i \rangle + \|e_i\|^2,$$

and, similarly,

$$\|x - e_i\|^2 = \|x\|^2 - 2\langle x, e_i\rangle + \|e_i\|^2.$$

We know from a) that $\|x\| = \|y\|$, so we conclude from the previous formulas and b),

$$x_i = \langle x, e_i\rangle = \langle y, e_i\rangle = y_i,$$

i.e., all components of $x$ and $y$ coincide. This implies that $x = y$.

Thus we have $k = \mathrm{id}$ and the proof is finished. □

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**Lemma 1.7.** *Assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry with $g(0) = 0$ and $g(e_i) = v_i$. Then $v_1, v_2, \dots, v_n$ are an orthonormal base of $\mathbb{R}^n$.*

*Proof.* We have to show that $\langle v_i, v_j\rangle = \delta_{ij}$.

a) $\|v_i\| = d(v_i, 0) = d(g(e_i), g(0)) = d(e_i, 0) = \|e_i\| = 1$.

b) $\|v_i - v_j\| = d(v_i, v_j) = d(g(e_i), g(e_j)) = d(e_i, e_j) = \|e_i - e_j\|$.

c) We assume that $i \neq j$. Squaring the left hand side of b) yields:

$$\|v_i - v_j\|^2 = \langle v_i - v_j, v_i - v_j\rangle = \|v_i\|^2 - 2\langle v_i, v_j\rangle + \|v_j\|^2 = 2 - 2\langle v_i, v_j\rangle,$$

by using a). Squaring the right hand side of b) yields, similarly,

$$\|e_i - e_j\|^2 = \|e_i\|^2 - 2\langle e_i, e_j\rangle + \|e_j\|^2 = 2.$$

Comparing both sides yields the required result

$$\langle v_i, v_j\rangle = 0.$$

□

**Corollary 1.8.** *Assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry with $g(0) = 0$ and $v_i = g(e_i)$. Then $A = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} \in O(n)$ and $g(x) = Ax$.*

*Proof.* Since $\langle v_i, v_j\rangle = v_i^\top v_j = \delta_{ij}$, we have $A^\top A = \mathrm{Id}$, i.e., $A \in O(n)$. Since $h(x) = Ax$ is an isometry with $h(0) = 0$ and $g(e_i) = v_i = h(e_i)$, we have $g = h$, by Lemma 1.6. □

*Proof of Theorem 1.5.* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry and $b = f(0)$. Then $g(x) = f(x) - b$ is also an isometry (since it is the composition $t_{-b} \circ f$ of the two isometries $f$ and $t_{-b}(x) = x - b$). We have $g(0) = 0$ and, thus, by Corollary 1.8, $g(x) = Ax$ with $A \in O(n)$. This implies that $f(x) = g(x) + b = Ax + b$. □

## 1.2 Classification of isometries of $\mathbb{R}^2$

Next, we want to classify isometries of $\mathbb{R}^2$. Let us first look at concrete examples:

**Examples.** *a)* translations*: $t_a(x) = x + a$*

*b)* rotations about origin*: $r_\alpha(x) = R_\alpha x$, $R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is counter clockwise rotation about origin by angle $\alpha$*

*c)* general rotations *about $z$: $r_{\alpha,z}(x) = R_\alpha(x - z) + z$*

*d)* reflection *along a line $l$: $s_l$. The set of fixed points of $s_l$ is the line $l$. Assume first that $l$ is a line through the origin, given by $l = \mathbb{R}v$ and $w \perp v$ and $\|v\| = \|w\| = 1$. If $x = \alpha v + \beta w$, then*

$$s_l(x) = \alpha v - \beta w = x - 2\beta w = x - 2\langle x, w\rangle w.$$

*If $l$ is a general line $l = w + V$ with $V$ equals a line through the origin, then $s_l(x) = s_V(x - w) + w$.*

*e)* glide reflection*: let $l = w + V$ be a line, $V$ be a parallel line through the origin and $a \in V$. The glide reflection $s_{l,a}$ is then defined to be*

$$s_{l,a} = s_l \circ t_a.$$

*Claim: $s_l \circ t_a = t_a \circ s_l$.*

*Proof: Using the fact that $s_V$ is a linear map and that $a$ lies in the fixed point set of $s_V$, we have*

$$\begin{aligned} s_l \circ t_a(x) &= s_l(x+a) = s_V(x+a-w) + w = s_v(x-w) + s_V(a) + w \\ &= s_v(x-w) + w + a = s_l(x) + a = t_a \circ s_l(x). \end{aligned}$$

**Theorem 1.9.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry different from the identity. Then $f$ is a translation $t_a$, a general rotation $r_{\alpha,z}$, a reflection $s_l$ along a line $l$, or a glide reflection $s_{l,a}$.*

**Definition 1.10.** *An isometry $f(x) = Ax + b$ of $\mathbb{R}^n$ is called* orientation preserving *or* orientation reversing, *if* $\det A = 1$ *or* $\det A = -1$.

*Proof of Theorem 1.9.* Let $f(x) = Ax + b$. Then the column vectors of $A$ are an orthonormal base. Every unit vector is of the form $\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}^\top$ for $\alpha \in [0, 2\pi)$ and a second orthogonal unit vector is either $\begin{pmatrix}-\sin\alpha & \cos\alpha\end{pmatrix}^\top$ or $\begin{pmatrix}\sin\alpha & -\cos\alpha\end{pmatrix}^\top$. Thus

$$A + R_\alpha = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix} \quad \text{or } A = S_\alpha = \begin{pmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{pmatrix}.$$

Let $f(x) = R_\alpha x + b$, i.e., an orientation preserving isometry, $\alpha \in [0, 2\pi)$. If $\alpha = 0$ then $b \neq 0$ (since $f \neq \mathrm{id}$) and $f(x) = x + b = t_b(x)$. If $\alpha \in (0, 2\pi)$ then

$$\det(I - A) = \det\begin{pmatrix}1-\cos\alpha & \sin\alpha \\ -\sin\alpha & 1-\cos\alpha\end{pmatrix} = (1-\cos\alpha)62 + \sin^2\alpha = 2(1-\cos\alpha) \neq 0.$$

Hence $(I - A)z = b$ has a unique solution $z \in \mathbb{R}^2$ and

$$r_{\alpha,z}(x) = R_\alpha(x - z) + z = R_\alpha x + (I - A)z = R_\alpha x + b = f(x).$$

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Let $f(x) = S_\alpha x + b$, i.e., an orientation reversing isometry. Since

$$\det(I - A) = \det\begin{pmatrix}1-\cos\alpha & -\sin\alpha \\ -\sin\alpha & 1+\cos\alpha\end{pmatrix} = 1 - \cos^2\alpha - \sin^2\alpha = 0,$$

$V = \ker I - A$ is one dimensional (we never have $I - A = 0$). If $b = 0$ then $f = S_\alpha = s_V$ (since all vectors in $V$ are fixed by $f$ and a vector $w \perp V$ must be mapped to $-w$ because of $f \neq \mathrm{id}$). If $b \neq 0$ then $b$ can be written as

$$b = 2w + v \quad \text{with } w \perp V,\ v \in V.$$

Note that $S_\alpha w = -w$, $S_\alpha v = v$. Let $l = w + V$. Then

$$\begin{aligned} s_{l,v}(x) &= s_l(x) + v = s_V(x - w) + w + v = s_V(x) + (I - s_V)(w) + v \\ &= s_V(x) + 2w + v = S_\alpha x + b = f(x), \end{aligned}$$

i.e., $f$ is a glide reflection. □

**Definition 1.11.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map. A point $x \in \mathbb{R}^n$ is called* fixed point *of $f$ if*

$$f(x) = x.$$

Let us investigate on the fixed points of isometries in $\mathbb{R}^2$:

(a) translations $t_a$, $a \neq 0$, have no fixed points at all, since $x = t_a(x) = x + a$ is never fulfilled.

(b) let $r_{\alpha,z}$ be a rotation with $\alpha(0, 2\pi)$. Then

$$\begin{aligned} & r_{\alpha,z}(x) = x \\ \Leftrightarrow\quad & R_\alpha(x - z) = x - z \\ \Leftrightarrow\quad & (I - R_\alpha)(x - z) = 0 \\ \Leftrightarrow\quad & x - z = 0 \quad \text{since } \det(I - R_\alpha) \neq 0 \end{aligned}$$

This shows that $z$ is the only fixed point of $r_{\alpha,z}$.

(c) the fixed points of a reflection $s_l$ along a line $l$ is obviously precisely the line $l$

(d) finally, we consider a glide reflection $s_{l,a}$, $a \neq 0$. Let $V$ be a line through the origin and parallel to $l$, i.e., $l = w + V$ for an appropriate vector $w \in \mathbb{R}^2$. Then

$$\begin{aligned} & s_{l,a}(x) = x \\ \Leftrightarrow\quad & s_l(x) + a = x \\ \Leftrightarrow\quad & a = x - s_l(x), \end{aligned}$$

But one easily sees that $x - s_l(x)$ is orthogonal to $V$, whereas $a \neq 0$ is parallel to $V$. This is a contradiction. So glide reflections don't have fixed points.

**Lemma 1.12.** *Let $f(x) = Ax + b$ and $g(x) + Cx + d$ be two isometries of $\mathbb{R}^n$. Then*

$$(f \circ g)(x) = ACx + e, \quad (g \circ f)(x) = CAx + f$$

*with suitable vectors $e, f \in \mathbb{R}^n$. In particular, the composition of two orientation preserving or reversing isometries is orientation preserving and the composition of an orientation preserving isometry with an orientation reversing is orientation reversing.*

*Proof.* Straighforward. □

**Remark 1.** *We have*

$$\begin{aligned} R_\alpha R_\beta &= R_{\alpha+\beta} \\ S_\alpha R_\beta &= S_{\alpha-\beta}, \qquad R_\alpha S_\beta = S_{\alpha+\beta} \\ S_\alpha S_\beta &= R_{\alpha-\beta}, \quad \textit{in particular } S_\alpha^{-1} = S_\alpha \end{aligned}$$

## 1.3 Conjugation of isometries of $\mathbb{R}^2$

Next, we look at conjugations of isometries in $\mathbb{R}^2$:

**Theorem 1.13.** *Let $f(x) = Ax + b \in I(\mathbb{R}^2)$. Then*

$$\begin{aligned} f \circ t_a \circ f^{-1} &= t_{Aa} \\ f \circ r_{\alpha,z} \circ f^{-1} &= r_{\det A \cdot \alpha, f(z)} \\ f \circ s_{l,a} \circ f^{-1} &= s_{f(l),Aa} \end{aligned}$$

*Proof.* Note that $f^{-1}(x) = A^{-1}x - A^{-1}b$. Then

$$\begin{aligned} (f \circ t_a \circ f^{-1})(x) &= f \circ t_a(A^{-1}x - A^{-1}b) = f(A^{-1}x - A^{-1}b + a) \\ &= A(A^{-1}x - A^{-1}b + a) + b = x + Aa = t_{Aa}(x), \end{aligned}$$

proving the first identity.

$f \circ r_{\alpha,z} \circ f^{-1}$ is orientation preserving, by Lemma 1.12, and has fixed point $f(z)$:

$$(f \circ r_{\alpha,z} \circ f^{-1})(f(z)) = f \circ r_{\alpha,z}(z) = f(z),$$

thus is a rotation about $f(z)$ by the Classification Theorem 1.9. We distinguish two cases:

a) $f(z) = R_\beta z + b$. Then, by Lemma 1.12,

$$(f \circ r_{\alpha,z} \circ f^{-1})(x) = R_\beta R_\alpha R_{-\beta} x + d = R_\alpha x + d$$

for a suitable $d \in \mathbb{R}^2$, i.e., $f \circ r_{\alpha,z} \circ f^{-1} = r_{\alpha,f(z)}$.

b) $f(z) = S_\beta z + b$. Then 20 October 2008

$$(f \circ r_{\alpha,z} \circ f^{-1})(x) = S_\beta R_\alpha S_\beta x + d = R_{-\alpha} x + d$$

for a suitable $d \in \mathbb{R}^2$, i.e., $f \circ r_{\alpha,z} \circ f^{-1} = r_{-\alpha,f(z)}$.

Finally, we first prove $f \circ s_l \circ f^{-1} = s_{f(l)}$: By Lemma 1.12, $f \circ s_l \circ f^{-1}$ is orientation reversing and fixing the line $f(l)$, since for $x \in f(l)$ we have $f^{-1}(x) \in l$ and:

$$f \circ s_l \circ f^{-1}(x) = f \circ s_l(f^{-1}(x)) = f(f^{-1}(x)) = x.$$

But there is only one such isometry, by the Classification Theorem 1.9, namely, $s_{f(l)}$. This implies

$$f \circ s_{l,a} \circ f^{-1} = f \circ s_l \circ t_a \circ f^{-1} = (f \circ s_l \circ f^{-1}) \circ (f \circ t_a \circ f^{-1}) = s_{f(l)} \circ t_{Aa} = s_{f(l),Aa}.$$

□

**Definition 1.14.** *Let $G$ be a group and $X$ be a set. An* action *of $G$ on $X$ is a map which assigns to every $g \in G$ a map $T_g : X \to X$ such that*

$$T_e = \mathrm{Id}_X, \qquad T_{g_1 \cdot g_2} = T_{g_1} \circ T_{g_2},$$

*for all $g_1, g_2 \in G$ ($e$ equals the identity element of $G$). An action is called* transitive, *if for every pair $x, y \in X$ there exists a $g \in G$ such that $T_g x = y$.*

Note that, in a group action, we obviously have $T_{g^1} = (T_g)^{-1}$, since

$$T_g \circ T_{g^{-1}} = T_{g \cdot g^{-1}} = T_e = \mathrm{Id}_X.$$

**Examples.** *(a) The vector space $G = \mathbb{R}^n$ is a commutative group under addition. It acts on $X = \mathbb{R}^n$ via translations $G \ni a \mapsto t_a : \mathbb{R}^n \to \mathbb{R}^n$. The action is transitive, since for $x, y \in X = \mathbb{R}^n$ and $a = y - x \in G = \mathbb{R}^n$ we have*

$$t_a(x) = x + a = x + (y - x) = y.$$

*(b) The matrix group $G = O(n)$ is a group under matrix multiplication. An action on $X = \mathbb{R}^n$ is: $G \ni A \mapsto R_A : \mathbb{R}^n \to \mathbb{R}^n$ with*

$$R_A(x) = Ax \qquad \forall\, x \in \mathbb{R}^n.$$

*Obviously, $R_I = \mathrm{Id}_{\mathbb{R}^n}$ and*

$$(R_A \circ R_B)(x) = A(Bx) = (AB)x = R_{A \cdot B}(x).$$

*This action is not transitive since $0 \in \mathbb{R}^n$ cannot be mapped to any other point in $\mathbb{R}^n$ via transformations $R_A$.*

**Theorem 1.15.** *Two elements of $I(\mathbb{R}^2)$ are conjugate if and only if one of the following statements is true:*

- *(a) both elements are the identity*
- *(b) both elements are translations by non-zero vectors of the same length*
- *(c) both elements are general rotations by angles in $[-\pi, \pi)$ of the same non-zero absolute value*
- *(d) both elements are reflections*
- *(e) both elements are glide reflections with the same non-zero glide distance*

*(Note that the* glide distance *of a glide reflection $s_{l,a}$ is the value $|a| > 0$.)*

*Proof.* It follows from Theorem 1.13 that if two isometries in $I(\mathbb{R}^2)$ are conjugate then they both belong to the same class (a)-(e). It remains to show that two isometries of the same class are conjugate: class (a) is trivial.

class (b): $t_a$ and $t_b$ with $\|a\| = \|b\|$. Obviously, there exists an $\alpha \in [0, 2\pi)$ such that $b = R_\alpha a$. If $f(x) = R_\alpha x$ then

$$f \circ t_a \circ f^{-1} = t_{Aa} = t_b.$$

class (c): $r_{\alpha,z}$ and $r_{\beta,w}$ with $\alpha, \beta \in [-\pi, \pi)$. If $\alpha = \beta$, choose $f(x) = x + (w - z)$, if $\alpha = -\beta$, choose $f(x) = S_0(x - z) + w$ with $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\det S_0 = -1$ and

$$f \circ r_{\alpha,z} \circ f^{-1} = r_{\beta,f(z)} = r_{\beta,w}.$$

class (d): $s_l$ and $s_{l'}$. If $l$ and $l'$ are parallel, then $l' = t_b(l)$ and if $l$ and $l'$ intersect in $z$, then $l' = r_{\alpha,z}(l)$ for suitable $\alpha \in [0, \pi)$. Choose $f = t_b$ or $f + r_{\alpha,z}$, respectively. Then

$$f \circ sl \circ f^{-1} = s_{f(l)} = s_{l'}.$$

class (e): $s_{l,a}$ and $s_{l',a'}$ with $\|a\| = \|a'\|$. If $l$ and $l'$ are parallel, then $l' = t_b(l)$ and $l = t_c(V)$ with $V$ a line through the origin, parallel to $l$ and $l'$, and $a' = \pm a$. To $a' = \pm a$, choose

$$f = t_{b+c} \circ (\pm \mathrm{Id}_{\mathbb{R}^2}) \circ t_{-c},$$

respectively. Then $f(l) = t_{b+c} \circ (\pm \mathrm{Id})(V) = t_{b+c}(V) = l'$ and

$$f \circ s_{l,a} \circ f^{-1} = s_{f(l),\pm a} = s_{l',a'}.$$

If $l$ and $l'$ intersect in $z$, then $l' = r_{\alpha,z}(l)$ for suitable $\alpha \in [0, \pi)$. Then $a' = \pm R_\alpha a$. If $a' = R_\alpha a$ then choose $f = r_{\alpha,z}$, if $a' = -R_\alpha a = R_{\alpha+\pi} a$ then choose $f = r_{\alpha+\pi,z}$. In the second case, we have $f(l) = r_{\alpha+\pi,z}(l) = l'$. Then

$$f \circ s_{l,a} \circ f^{-1} = s_{f(l),a'} = s_{l',a'}.$$

□

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## 1.4 Symmetry groups

**Definition 1.16.** *Let $S \subset \mathbb{R}^n$ be a set. The* symmetry group *of $S$ is given by*

$$\Gamma(S) = \{f \in I(\mathbb{R}^n) \mid f(S) = S\}.$$

*Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a function. The* symmetry group *of $h$ is given by*

$$\Gamma(h) = \{f \in I(\mathbb{R}^n) \mid h \circ f = h\}.$$

**Remark 2.** *The definition of a symmetry group of a function is more general than the definition of a symmetry group of a set. If $S$ is a set, we can choose the characteristic function*

$$h(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases}$$

*We then obtain $\Gamma(S) = \Gamma(h)$. But $h$ can encode more information: brightness, colour, ... of a pattern (note that $h$ is even allowed to be vector valued).*

**Examples.** *(a)* windmill *$S$: a symmetry $f$ must fix the origin $0$ and must map the vertex $x$ to one of the $8$ vertices. This can be achieved by $r_{k\pi/4}$, $k \in \{0, 1, \dots, 7\}$. Then $r_{-k\pi/4} \circ f$ fixes the origin and $x$, i.e., it is either the identity or the reflection along the horizontal line. But the latter is not in the symmetry group $\Gamma(S)$. Thus*

$$\Gamma(S) = \{r_{k\pi/4} \mid k \in \{0, 1, \dots, 7\}\}.$$

*(b)* heptagon $s$*: a symmetry must fix the origin* $0$ *and must map the vertex* $x$ *to one of the* $7$ *vertices. This can be achieved by* $r_{2k\pi/7}$, $k \in \{0, \ldots, 6\}$. *Then* $r_{-2k\pi/7} \circ f$ *fixes the origin and* $x$*, i.e., is either the identity or reflection along the vertical line* $l$*. The latter is in* $\Gamma(S)$*. Thus we have*

$$\begin{aligned} \Gamma(S) &= \{r_{2k\pi/7} \mid k \in \{0, \ldots, 7\}\} \cup \{r_{2k\pi/7} \circ s_l \mid k \in \{0, \ldots, 7\}\} \\ &\cong D_7 \text{ (dihedral group)} \end{aligned}$$

*(c)* infinite net $S$*: Any symmetry* $f$ *must map* $0$ *to some node of the form* $mv_1 + nv_2$, $m, n \in \mathbb{Z}$. *This can be achieved by* $t_{mv_1+nv_2}$*, and hence* $t_{-(mv_1+nv_2} \circ f$ *fixes the origin, i.e., is either a rotation of a reflection through the origin. Since* $v_1$ *is not perpendicular to* $v_2$ *and* $\|v_1\| \neq \|v_2\|$*, no reflection is in* $\Gamma(S)$*. So* $t_{-(mv_1+nv_2)} \circ f$ *is either the identity of a rotation by* $\pi$ *and*

$$\Gamma(S) = \{t_m v_1 + nv_2\} \circ r_{k\pi} \mid m, n \in \mathbb{Z}, k \in \{0, 1\}\}.$$

*(d)* zig-zag pattern $S$*: Any symmetry* $f$ *must map* $0$ *to some node, but the nodes* $mv_1 + nv_2$ *are only the downward pointing nodes. An upward pointing node can be reached from* $0$ *by a glide reflection along the horizontal line* $l$ *and glide vector* $\frac{v_1}{2}$*:* $s_{l,v_1/2}$*. So either* $t_{-(mv_1+nv_2} \circ f$ *or* $t_{-(mv_1+nv_2} \circ s_{l,v_1/2}6{-1} \circ f$ *is fixing the origin and thus is either the identity or the reflection* $s_{l'}$*, where* $l'$ *is the vertical axis. This implies that*

$$\begin{aligned} \Gamma(S) &= \text{group generated by } t_{v_1}, t_{v_2}, s_{l,v_1/2}, s_{l'} \\ &= \langle t_{v-1}, t_{v_2}, s_{l,v_1/2}, s_{l'} \rangle. \end{aligned}$$

*Note that these four isometries are not independent. We have, e.g.,* $t_{v_1} = s_{l,v_1/2}^2$.

A natural goal would be to classify all symmetry groups of $I(\mathbb{R}^n)$, at least up to isomorphism. But this goal is too ambitious. Instead, we try to understand all *discrete* symmetry groups of $\mathbb{R}^2$ a bit better.

**Definition 1.17.** *A subgroup* $\Gamma \subset I(\mathbb{R}^n)$ *is called* discrete *if, for any* $x_0 \in \mathbb{R}^n$ *and any bounded set* $B \subset \mathbb{R}^n$*, the set*

$$\{f \in \Gamma \mid f(x_0) \in B\}$$

*is finite. A discrete subgroup* $\Gamma \subset I(\mathbb{R}^n)$ *is called* uniform*, if there is a compact aet* $K \subset \mathbb{R}^n$ *such that*

$$\bigcup_{f \in \Gamma} f(K) = \mathbb{R}^n.$$

*A discrete uniform subgroup of* $I(\mathbb{R}^n)$ *is also called a* crystallographic group*.*

**Examples.** *(a) The group* $\Gamma = \{t_a \mid a \in \mathbb{Q}^2\} \subset I(\mathbb{R}^2)$ *is not discrete since, for* $x = 0 \in \mathbb{R}^2$ *and* $B = \{z \in \mathbb{R}^2 \mid \|z\| \leq 1\}$ *we have*

$$|\{f \in \Gamma \mid f(0) \in B\}| = |\{a \in \mathbb{Q}^2 \mid \|a\| \leq 1\}| = \infty.$$

(b) *The groups* $\Gamma = \{t_a \mid a \in \mathbb{Z}^2\}$ *or* $\Gamma = \{t_a \mid a \in \mathbb{Z} \times \{0\}\}$ *are both discrete. The first group is uniform since, for* $K = [0,1] \times [0,1]$,

$$\bigcup_{f \in \Gamma} f(K) = \bigcup_{a \in \mathbb{Z}^2} a + K = \mathbb{R}^2,$$

*but the second group is not uniform since every compact set* $K \subset \mathbb{R}^2$ *is contained in a large enough square* $Q = [-n,n] \times [-n,n]$ *with* $n \geq 1$ *and*

$$\bigcup_{f \in \Gamma} f(K) \subset \bigcup_{f \in \Gamma} f(Q) = \mathbb{R} \times [-n,n] \neq \mathbb{R}^2.$$

**Remark 3.** *For discreteness of* $\Gamma$ *it is enough to check the following: For every ball* $B_r = \{z \in \mathbb{R}^n \mid \| \ \| \leq r\}$, *we have*

$$|\{f \in \Gamma \mid f(0) \in B_r\}| < \infty.$$

*This fact is proved in Exercise 5.*

## 1.5 Translation subgroup and derived group

**Definition 1.18.** *Let* $\Gamma \subset I(\mathbb{R}^n)$ *be a subgroup. The* translation subgroup $T(\Gamma)$ *is defined as*

$$T(\Gamma) = \{t_a \mid a \in \mathbb{R}^n\}$$

*and is isomorphic to* $L = \{a \in \mathbb{R}^n \mid t_a \in \Gamma\}$. *The* derived group $\Gamma'$ *is defined as*

$$\Gamma' = \{f'(x) = Ax \mid f(x) = Ax + b \in \Gamma\}.$$

Both groups $T(\Gamma)$ and $\Gamma'$ play an important role in *Crystallography.*

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**Lemma 1.19.** *Let* $\Gamma \subset I(\mathbb{R}^2)$ *be a subgroup.*

(a) *If* $t_a \in T(\Gamma)$, $a \neq 0$, *then*

$$\{t_{f'(a)} \mid f \in \Gamma\} \subset T(\Gamma).$$

(b) *If* $T(\Gamma) = \{\mathrm{Id}_{\mathbb{R}^2}\}$, *then all* $f \in \Gamma$ *have a common fixed point* $x_o \in \mathbb{R}^2$.

*Proof.* (a) Let $f(x) = Ax + b$. Then, by Theorem 1.13,

$$f \circ t_a \circ f^{-1} = t_{Aa} = t_{f'(a)} \in T(\Gamma).$$

(b) $T(\Gamma) = \{\mathrm{Id}_{\mathbb{R}^2}\}$ implies that $\Gamma$ contains no glide reflection $s_{l,a}$, $a \neq 0$, since otherwise

$$s_{l,a}^2 = (t_a \circ s_l) \circ (s_l \circ t_a) = t_{2a} \in T(\Gamma).$$

Assume that $f_1, f_2 \in \Gamma$ have *no common fixed point.* If both are rotations $r_{\alpha,z}$ and $r_{\beta,w}$ with $\alpha, \beta \in (0, 2\pi)$ and $zneqw$, then

$$r_{\alpha,z} \circ r_{\beta,w} \neq r_{\beta,w} \circ r_{\alpha,z}, \tag{1}$$

since otherwise we would have

$$r_{\beta,w}(r_{\alpha,z}(w)) = r_{\alpha,z}(w),$$

i.e., $r_{\alpha,z}(w)$ would be fixed point of $r_{\beta,w}$, i.e.,

$$r_{\alpha,z}(w) = w,$$

i.e., $w$ would be fixed point of $r_{\alpha,z}$, i.e., $z = w$, a contradiction. Since

$$r_{\alpha,z} \circ r_{\beta,w} \circ (r_{\beta,w} \circ w_{\alpha,z})^{-1}(x) = Ax + b$$

has trivial linear part by Lemma 1.12, this isometry is a translation. This translation is non-trivial because of (1), which contradicts to $T(\Gamma) = \{\mathrm{Id}_{\mathbb{R}^2}\}$. Consequently, all rotations in $\Gamma$ have a common fixed point.

Let $r_{\alpha,z}$ with $\alpha \in (0, 2\Pi)$ and $s_l$ be in $\Gamma$ with $z \neq l$. Then we have $s_l(z) \neq z$. By Theorem 1.13, we obtain

$$s_l \circ r_{\alpha,z} \circ s_l^{-1} = r_{-\alpha, s_l(z)} \in \Gamma,$$

but then $\Gamma$ would contain two rotations with different fixed points, which was ruled out before. Therefore, if $\Gamma$ contains a rotation with fixed point $z$, then all reflections $s_l$ must satisfy $z \in l$. It remains to consider the case when $\Gamma$ doesn't contain any rotations at all, i.e., that all non-trivial elements of $\Gamma$ are reflections. If there are two different reflections $s_l, s_{l'} \in \Gamma$, then $s_l \circ s_{l'}$ is either a non-trivial translation (if $l$ and $l'$ are parallel) or a non-trivial rotation (if $l$ and $l'$ intersect). But these possibilities are ruled out under the condition that the non-trivial elements of $\Gamma$ are only reflections. So in this case we must either have $\Gamma = \{\mathrm{Id}_{\mathbb{R}^2}\}$ or $\Gamma = \{\mathrm{Id}_{\mathbb{R}^2}, s_l\}$, and in both cases all isometries of $\Gamma$ have a common fixed point. ☐

**Corollary 1.20.** *Let* $\Gamma \subset I(\mathbb{R}^2)$ *be a* discrete *subgroup. Then*

*(a)* $T(\Gamma)$ *is generated by linearly independent vectors, hence is isomorphic to* $\{0\}$, $\mathbb{Z}$ *or* $\mathbb{Z}^2$.

*(b)* $\Gamma'$ *is finite.*

*(c)* $\Gamma$ *is finite if and only if* $T(\Gamma) = \{\mathrm{Id}_{\mathbb{R}^2}\}$.

*Proof.* We skip the proof of (a).

We first assume that $T(\Gamma) = \{\mathrm{Id}_{\mathbb{R}^2}\}$. Then all $f \in \Gamma$ have a common fixed point $x_0 \in \mathbb{R}^2$. Let $x_1 \in \mathbb{R}^2$ with $d(x_1, x_0) = 1$ and $B_1(x_0) := \{y \in \mathbb{R}^2 \mid d(y, x_0) \leq 1\}$. Then $f(x_1) \in B_1(x_0)$ for all $f \in \Gamma$ and, by discreteness,

$$|\Gamma| = |\{f \in \Gamma \mid f(x_1) \in B_1(x_0)\}| < \infty.$$

Consequently, we also have $|\Gamma'| < \infty$.

Now, assume that $t_a \in T(\Gamma)$, $a \neq 0$. Then $\{t_{ka} \mid k \in \mathbb{Z}\} \subset T(\Gamma)$, and $T(\Gamma)$ is infinite. This implies that $\Gamma$ is also infinite. Since

$$\{t_{f'(a)} \mid f \in \Gamma\} \subset T(\Gamma),$$

we conclude from the discreteness of $T(\Gamma)$ that $\{f'(a) \mid f \in \Gamma\} \subset B_{\|a\|}(0) = \{y \in \mathbb{R}^2 \mid d(y, 0) \leq \|a\|\}$ is finite. Since there are at most two *linear* isometries ($f$ is a linear isometry if $f(x) = Ax$ without translation part), namely a particular rotation about 0 and a reflection $s_l$ with $0 \in l$, which map $a$ to $f'(a)$, $\Gamma'$ is also finite. ☐

**Theorem 1.21.** *If a discrete group $\Gamma \subset I(\mathbb{R}^n)$ is infinite, then $\Gamma'$ is isomorphic to $C_k$ (the cyclic group of order $k$) or $D_k$ (the dihedral group of order $2k$) with $k \in \{1, 2, 3, 4, 6\}$.*

*Proof.* We skip the proof of the fact that discreteness of $\Gamma$ implies $\Gamma' \cong C_k$ or $D_k$ for some $k \in \mathbb{N}$ and prove only the restriction of $k$ to $\{1, 2, 3, 4, 6\}$. We assume $k \neq 1$.

Let $r = r_{e\pi/k} \in \Gamma'$ be a rotation by a minimal angle $\alpha \in (0, 2\pi)$ and $t_a \in T(\Gamma)$ be a non-zero translation with minimal $\|a\| > 0$. Then $t_{r(a)-a} \in T(\Gamma)$, by Lemma 1.19 (a), and $r(a) - a \neq 0$, since $k \neq 1$. Now,

$$\|r(a) - a\|^2 = \|r(a)\|^2 - 2\langle r(a), a\rangle + \|aVert^2 = 2\|a\|^2(1 - \cos\frac{2\pi}{k}) \geq \|a\|^2,$$

i.e., $cos\frac{2\pi}{k} \leq \frac{1}{2}$, $k \leq 6$. Now, assume that $k = 5$. Then

$$0 < \|r^2(a) + a\|^2 = 2\|a\|^2(1 + \cos\frac{4\pi}{5}) < \|a\|^2$$

and $\mathrm{id}_{\mathbb{R}^2} \neq t_{r^2(a)+a} \in T(\gamma)$, contradicting to the minimality of $\|a\|$. ☐

The fact that rotations in a discrete group of isometries can only have orders $2, 3, 4, 6$ holds in $\mathbb{R}^n$ for dimensions $n = 2$ and $n = 3$, and is called the *Crystallographic Restriction Theorem.* Crystallographic groups in $I(\mathbb{R}^2)$ are also called *wallpaper groups* and you can find more about them at the webpage `http://en.wikipedia.org/wiki/Wallpaper group`. There are 17 distinct wallpaper groups, up to isomorphism.

The classification of crystallographic groups (i.e., discrete and uniform subgroups of $I(\mathbb{R}^n)$) is of practical importance in dimension 3. There are 219 different crystallographic groups in dimension 3, up to isomorphism (see, e.g., the webpage `http://en.wikipedia.org/wiki/Space group`). Let us finally mention (without proofs) the famous Bieberbach theorems:

**Theorem 1.22** (Bieberbach (1912))**.** *Let $\Gamma \subset I(\mathbb{R}^n)$ be a crystallographic group. Then $T(\Gamma)$ is a* normal *subgroup of $\Gamma$ of* finite index *and a* lattice *(i.e., of the form $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ with $v_1, \ldots, v_n$ linearly independent).*

**Theorem 1.23** (Bieberbach (1912))**.** *For every dimension $n \in \mathbb{N}$, there is only a* finite number of isomorphism classes *of crystallographic groups $\Gamma \subset I(\mathbb{R}^n)$. Two crystallographic groups are isomorphic if and only if they are affine conjugate.*

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## 1.6 Fundamental domains and orbit spaces

Next, we introduce the important notions of *fundamental set* and *fundamental domain.*

**Definition 1.24.** *Let $G$ be a group acting on a set $X$. This defines an equivalence relation $\sim$ on $X$: We write $x_1 \sim x_2$ if there is an element $g \in G$ such that $x_2 = gx_1$. The equivalence classes*

$$[x] := \{x' \in X \mid x' \sim x\}$$

*are called* orbits *of the group action. A* fundamental set *$S$ is obtained by choosing one particular element in each orbit, i.e.*

$$|S \cap [x]| = 1 \qquad \textit{for every orbit } [x].$$

The existence of a fundamental set in general is guaranteed by the *axiom of choice*. But in many concrete cases a fundamental set can be chosen in an explicit way.

**Examples.** *(a) Let $G$ be the group generated by all reflections $s_n : \mathbb{R} \to \mathbb{R}$ at integer points $n \in \mathbb{Z}$. Note that $s_n(x) = -(x-n) + n = 2n - x$ and*

$$(s_{n+1} \circ s_n)(x) = s_{n+1}(2n - x) = (2n + 2) - (2n - x) = x + 2 = t_2(x).$$

*The orbits are given by*

$$[x] = \{x + 2n \mid n \in \mathbb{Z}\} \cup \{2n - x \mid n \in \mathbb{Z}\},$$

*in particular $[0] = 2\mathbb{Z}$ and $[1] = 2\mathbb{Z} + 1$ and a fundamental set is given by $S = [0, 1]$.*

*(b) Let $G$ be the group generated by all translations $t_n : \mathbb{R} \to \mathbb{R}$, $t_n(x) = x+n$ for $n \in \mathbb{Z}$. The orbits are given by $[x] = x + \mathbb{Z}$ and a fundamental set is given by $S = [0, 1)$.*

Often, we don't need these strict properties of a fundamental set. A set with somewhat weaker properties is presented in the next definition:

**Definition 1.25.** *An open connected domain $F \subset \mathbb{R}^n$ is called a* fundamental domain *for a discrete group $\Gamma \subset I(\mathbb{R}^n)$ if it satisfies the following conditions:*

*(a) $\bigcup_{g \in \Gamma} \overline{gF} = \mathbb{R}^n$, where $\overline{U}$ denotes the closure of $U \subset \mathbb{R}^n$.*

*(b) For all $g \in \Gamma$, $g \neq e$: $F \cap gF = \emptyset$.*

*(c) There are only finitely many $g \in \Gamma$ such that*

$$\overline{F} \cap \overline{gF} \neq \emptyset.$$

**Examples.** *(a) Let $v_1, v_2 \in \mathbb{R}^2$ be two linear independent vectors and $\Gamma = \{t_{nv_1 + mv_2} : \mathbb{R}^2 \to \mathbb{R}^2 \mid n, m \in \mathbb{Z}\}$. Then a natural fundamental domain is the open parallelogram*

$$F := \{t_1 v_1 + t_2 v_2 \mid t_1, t_2 \in (0, 1)\}.$$

*The picture one should have in mind is that the "tiles" $gF$, $g \in G$, tessellate the plane without overlapping.*

*(b) Let $\Gamma$ be isometry group of the honeycomb pattern $S \subset \mathbb{R}^2$. Let the origin be placed in the centre of a cell $C_0$ of this pattern. Then every $g \in \Gamma$ must map $C_0$ to a cell of the pattern. The subgroup $\Gamma_0 := \{g \in \Gamma \mid gC_0 = C_0\}$ is isomorphic to the dihedral group $D_6$. Choose $F$ to be the open triangle with the origin, a vertex of $C_0$ and a midpoint of an adjacent side of $C_0$ as its vertices. Then*

$$\bigcup_{g \in \Gamma_0} \overline{gF} = \overline{C_0},$$

*and since every cell of the pattern can be reached from $C_0$ by a translation of $\Gamma$, we have*

$$\bigcup_{g\ in\Gamma} \overline{gF} = \mathbb{R}^2,$$

*proving property (a). Obviously, $gF \cap F = \emptyset$ for all $g \in \Gamma_0$, since the dihedral group unfolds this triangle in the hexagon. This proves (b), since any group element mapping $C_0$ to a different cell, maps $F$ to a triangle disjoint to $F$. Finally, one checks that $F$ meets 16 neighboring triangles, proving property (c).*

Let $G$ act on a set $X$. Then there is an obvious bijection between the orbits $[x] subset X$ and points of a fundamental set $S$. If we denote the *orbit space* by $X/G$, we thus have a $1:1$-relation between the elements in $X/G$ (the orbits) and the points of $S$. But two points $x, y$ in $S$ might be quite far apart even though the orbits $[x]$ and $[y]$ are close to each other. Let us look at an example: $X = \mathbb{R}$ and $G = \{t_n \mid n \in\}$, $S = [0, 1)$. Then the orbits $[0] = \mathbb{Z}$ and $[0.99] = 0.99 + \mathbb{Z} = \mathbb{Z} - 0.01$ are very close, even though the points $0, 0.99 \in S$ are far apart. To remedy this, one should think of $X/G$ as the closed interval $[0, 1]$ with the points 0 and 1 identified. Topologically, this would coincide with a circle $S^1$.

Let us, finally, discuss two other 2-dimensional examples:

**Examples.** *(a)* **Torus:** *Let $\Gamma = \{t_{ne_1+me_2} \mid n, m \in \mathbb{Z}\}$ acting on $\mathbb{R}^2$. A fundamental set is $S = [0, 1)\times[0, 1)$. Since the orbits $(x, 0)+\mathbb{Z}^2$ and $(x, 0.99)+\mathbb{Z}^2$ are very close as well as the $(0, y) + \mathbb{Z}^2$ and $(0.99, y) + \mathbb{Z}^2$, we should represent the orbit space $\mathbb{R}^2/\Gamma$ by the closed square $[0, 1]\times[0, 1]$ where we identify the lower and upper side and the left and right side, i.e. $(x, 0)$ is identified with $(x, 1)$ and $(0, y)$ is identified with $(1, y)$. These identifications imply that all four vertices $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$ are identified as one point. These identifications yield, topologically, a two-dimensional torus $T^2$ as the space representing the orbit space $\mathbb{R}^2/\Gamma$.*

*(b)* **Klein bottle:** *Let $\Gamma$ be generated by the elements $t_{e_2}$ and $s_{l,e_1}$, where $l$ is the horizontal axis. Note that we have $s^2_{l,e_1} = t_{2e_1}$. A fundamental domain is given by $F = (0, 1) \times (-1/2, 1/2)$. Straightforward considerations lead to the conclusion that the orbit space $\mathbb{R}^2/\Gamma$ should be seen as the closed square $[0, 1] \times [-1/2, 1/2]$ with the side identifications $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$. Again, all four vertices $(0, -1/2), (1, -1/2), (1, 1/2)$ and $(0, 1/2)$ are identified, but the topological surface now obtained is non-orientable and called the Klein bottle. This surface cannot be embedded into $\mathbb{R}^3$ (we need $\mathbb{R}^4$ for this), but it could be immersed into $\mathbb{R}^3$ with self-intersections. This surface is called* Klein bottle *after Felix Klein, who set up the concept that we should understand different geometries by studying the associated groups of these geometries. WE come back to this theme straight at the beginning of the next chapter.*

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# 2 Affine Geometry

## 2.1 Affine transformations and parallel projections

Let us start again with Klein's point of view:

- Euclidean geometry is based on a space $\mathbb{R}^n$ with the transformation group $I(\mathbb{R}^n)$ of isometries.

- Affine geometry, the topic of this chapter, is based on a space $\mathbb{R}^n$ with the transformation group $A(\mathbb{R}^n)$ of *affine transformations*, i.e.,

$$A(\mathbb{R}^n) := \{f(x) = Ax + b \mid A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n\}.$$

Note that we have $I(\mathbb{R}^n) \subset A(\mathbb{R}^n)$ and that $A(\mathbb{R}^n)$ is a group: if $f(x) = Ax + b$ and $g(x) = Cx + d$ then we have

$$\begin{aligned} (f \circ g)(x) &= A(Cx + d) + b = ACx + Ad + b, \\ f^{-1}(x) &= A^{-1}x - A^{-1}b. \end{aligned}$$

In the following we restrict our considerations entirely to two dimensions, i.e., $n = 2$.

While distances $d(x, y) = \|x - y\|$ are preserved under $I(\mathbb{R}^2)$, which are the geometric properties preserved by $A(\mathbb{R}^2)$? Certainly, no longer distances, as can be seen by the affine map $f(x) = 2x$.

We first introduce bijective maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, which are called *parallel projections*, and which will be later seen to be affine projections. These maps are defined by embedding domain and image of the map $f$ as different planes into $\mathbb{R}^3$. Such a higher dimension embedding can be used to prove elegantly highly non-trivial facts with only little use of 3-dimensional geometry. We will employ this method also very successfully when we study *Projective Geometry*.

**Example** (parallel projection)**.** *Represent two copies of $\mathbb{R}^2$ by two separate planes $\pi_1, \pi_2$ with their coordinate axes. Place the planes $\pi_1, \pi_2$ into $\mathbb{R}^3$. A map $f : \pi_1 \to \pi_2$ is defined via parallel rays (neither $\pi_1$ nor $\pi_2$ should be parallel to these rays so that each ray intersects both planes in uniquely determined points). Note that the so-defined map $f : \mathbb{R}^2 \to \mathbb{R}^2$ remains the same if we move $\pi_1$ or $\pi_2$ parallel along the rays. $f$ is obviously bijective and called a* parallel projection. *The inverse map $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is also a parallel projection (we obtain it by reversing the directions of all rays).*

A parallel projection $f : \pi_1 \to \pi_2$ is an *isometry* if $\pi_1$ and $\pi_2$ are parallel. In fact, every isometry can be realized by a parallel projection.

Next, we list and prove some fundamental geometric properties of parallel projections:

**Proposition 2.1.** *let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a a parallel projection. Then:*

*(a) $f$ maps straight lines to straight lines.*

*(b) $f$ maps parallel lines to parallel lines.*

*(c) $f$ preserves the ratios of lengths along a given straight line.*

*Proof.* (a) The rays through a straight line $l \subset \pi_1$ fill a plane $\Sigma \subset \mathbb{R}^3$. This plane intersects the non-parallel plane $\pi_2$ in a line $l' \subset \pi_2$, which coincides with the image $f(l)$.

(b) Let $l_1, l_2 \subset \pi_1$ be two parallel lines. The rays through $l_i$ fill a plane $\Sigma_i \subset \mathbb{R}^3$. Let $l_i' = f(l_i) = \Sigma_i \cap \pi_2$. If $l_1' \cap l_2' \sup\{P\} \subset \pi_2$, then $f^{-1}(P) \in \pi_1$

would lie in both planes $\Sigma_1, \Sigma_2$ and therefore also in $l_1 \cap l_2$. But $l_1 \cap l_2 = \emptyset$, because both lines are parallel. This is a contradiction and we conclude that $l_1' \cap l_2' = \emptyset$.

(c) Let $A, B, C$ be three different points on a line $l$ in $\pi_1$ and $A' = f(A)$, $B' = (B)$ and $C' = f(C)$ their images in $\pi_2$. We already know that $A', B', C'$ lie on a line, namely on $l' = f(l) \subset \pi_2$. We have to show that

$$\frac{d(A,B)}{d(A,C)} = \frac{d(A',B')}{d(A',C')}. \tag{2}$$

If both planes $\pi_1, \pi_2$ are parallel, then $f$ is an isometry and there is nothing to prove. Therefore, we assume that both planes are not parallel. By moving $\pi_2$, we can assume w.l.o.g. that $A$ coincides with $A'$ in $\mathbb{R}^3$. Then the two lines $l, l' \subset \mathbb{R}^3$ intersect in $A = A'$. If they coincide, there is nothing to prove. So let us assume that they don't coincide and that $A = A'$ is their only intersection point. Then $l, l' \subset \mathbb{R}^3$ span a plane, which we denote by $\Sigma$. $\Sigma$ contains all six points $A, B, C, A', B', C'$ as well as the lines $l, l'$. Since the pairs of points $B, B'$ and $C, C'$ can be connected by two parallel line segments in $\Sigma$ (since they are images under parallel rays), the triangles $\Delta ABB'$ and $\Delta ACC' = \Delta A'CC'$ are similar, i.e., one triangle is, up to congruence, a rescaled image of the other triangle. This immediately implies the desired equality (2). $\square$

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Next, we show that parallel projections are affine transformations:

**Proposition 2.2.** *Every parallel projection $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation, but not every affine transformation is a parallel projection.*

*Proof.* We first consider a parallel projection $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(0) = 0$. We prove that $f$ is linear:

Let $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^2$. The points $0, v, \lambda v$ lie on a line $l$ through the origin. They are mapped to the points $f(0) = 0, f(v), f(\lambda v)$ on a line $l'$. Since $f$ *preserves ratios along lines*, we must have $f(\lambda v) = \lambda f(v)$.

Let $v, w \in \mathbb{R}^2$. We can assume that $v$ and $w$ are linear independent, since otherwise one of the vectors is a multiple of the other, e.g., $w = \mu v$, and we can use the previous argument to show that

$$f(v+w) = f((1+\mu)v) = (1+\mu)f(v) = f(v)+\mu f(v) = f(v)+f(\mu v) = f(v)+f(w).$$

Linear independence of $v, w$ implies that $0, v, v+w, w$ are the vertices of a parallelogram in $\pi_1$. Since $f$ maps *parallel lines to parallel lines*, $f(0) = 0, f(v), f(v+w), f(w)$ must be the vertices of a parallelogram in $\pi_2$. Since $0, f(v), f(v) + f(w), f(w)$ is also a parallelogram in $\pi_2$, both parallelograms must be equal (three of the four vertices of a parallelogram determine the fourth). This implies that $f(v+w) = f(v) + f(w)$.

Since $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and invertible, we must have $f(x) = Ax$ with $A \in GL(2, \mathbb{R})$.

Now, assume that : $\mathbb{R}^2 \to \mathbb{R}^2$ is a parallel projection with $f(0) = b$. Then $g(x) = f(x) - b$ is also a parallel projection (by just readjusting the coordinate axes of $\pi_2$) satisfying $g(0) = 0$, so we have $g(x) = Ax$ with $A \in GL(2, \mathbb{R})$. This implies that $f(x) = Ax + b$, i.e., an affine transformation.

Finally, we convince ourselves that the affine transformation $f(x) = 2x$ cannot be realized as a parallel projection. Since $f(0) = 0$, a parallel projection representing $f$ could be set up that both planes $\pi_1$ and $\pi_2$ intersect in their origins. Obviously, both planes cannot be parallel, since then $f$ would be an isometry, which it isn't. Therefore, both planes intersect in a line $l$ through the origin. Vectors on this line in $\pi_1$ are mapped to vectors of the same length in $\pi_2$. But $f(x) = 2x$ does not preserve the length of any non-zero vector. ☐

Even though we cannot realize every affine transformation $f$ by a parallel projection, we can realize $f$ as the composition of two parallel projections. This implies that the set of parallel projections doesn't have a group structure (under composition) but that it is large enough to generate the group of affine transformations. This is the content of the next proposition:

**Proposition 2.3.** *Every affine transformation can be obtained as the composition of two parallel projections.*

*Proof.* (a) We first prove that an affine transformation $f(x) = Ax+b$ is uniquely determined by the images $f(0), f(e_1), f(e_2)$: Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$. Then $b = f(0)$ and

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} &= f(e_1) - b, \\ \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} &= f(e_2) - b. \end{aligned}$$

This means that we can reconstruct the affine transformation $f$ from $f(0), f(e_1), f(e_2)$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation with $f(0) = P, f(e_1) = Q, f(e_2) = R$. Below, we construct two parallel transformations $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\begin{aligned} &g_1(0) = P, \quad g_1(e_1) = Q, \quad g_1(e_2) = X \in \mathbb{R}^2, \\ &g_2(P) = P, \quad g_2(Q) = Q, \quad g_2(X) = R. \end{aligned}$$

Then the affine transformations $g_2 \circ g_1$ and $f$ coincide in the points $0, e_1, e_2$ and, therefore, are equal.

Place $\pi_1$ into $\mathbb{R}^3$ and a second plane $\pi_2$, not parallel to $pi_1$, intersecting $\pi_1$ in the origin, but not in the $x$-axis of $\pi_1$. Arrange the coordinate system of $\pi_2$ such that the origin of $\pi_1$ coincides with the point $P$ of $\pi_2$ and that $Q$ does not lie on the line $\pi_1 \cap \pi_2$ of intersection. Now, the line connecting $e_1 \in \pi_1$ with $Q \in \pi_2$ is not parallel to any of the two planes $\pi_1, \pi_2$ and defines a parallel projection $g_1$ satisfying $g_1(0) = P$, $g_1(e_1) = Q$ and $g_1(e_2) = X \in \pi_2$. Since $0, e_1, e_2$ don't lie on a common straight line, their images $P, Q, X$ under $g_1$ don't lie on a common straight line, either.

Now, we introduce a third plane $\pi_3$ in order to define $g_2$. We place $\pi_3$ in such a way that it is not parallel to $\pi_2$ and it intersects $\pi_2$ in the line through $P, Q \in \pi_2$. Choose the coordinate system of $\pi_3$ in such a way that $P, Q \in \pi_2$ are mapped to points in $\pi_3$ with the same coordinates. Therefore, we have $g_2(P) = P$ and $g_2(Q) = Q$, whatever parallel projection we consider between the planes $\pi_2$ and $\pi_3$. We know from above that $X \in \pi_2$ does not lie on the line of intersection $l = \pi_2 \cap \pi_3$. In order to know that the point with the coordinates of $R$ in $\pi_3$ lies not also in $l$, we use the fact that affine transformations map $0, e_1, e_2$ to three affine independent points, a fact, which we will prove later. Anticipating this result, we can conclude that the point $R \in \pi_3$ does not lie on $l$. Therefore, the line connecting $X \in \pi_2$ with $R \in \pi_3$ is not parallel two the two planes $\pi_2$ and $\pi_3$ and defines a parallel projection $g_2$ satisfying $g_2(P) = P$, $g_2(Q) = Q$ and $g_2(X) = R$, finishing the proof. □

We obtain as an immediate corollary:

**Corollary 2.4.** *Let $f(x) = Ax + b$ with $A \in GL(2,\mathbb{R})$ and $b \in \mathbb{R}^2$ be an affine transformation. Then:*

*(a) $f$ maps straight lines to straight lines.*

*(b) $f$ maps parallel lines to parallel lines.*

*(c) $f$ preserves the ratios of lengths along a given straight line.*

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## 2.2 Fundamental Theorem of Affine Geometry

Next, we leave the 3-dimensional geometry behind and use a little bit of matrix algebra in the arguments to follow. We first introduce the following important notion in higher dimensional space $\mathbb{R}^n$:

**Definition 2.5.** *The points $P_0, P_1, \ldots, P_k \in \mathbb{R}^n$ are called* affine independent*, if the vectors $P_1 - P_0, \ldots, P_k - P_0 \in \mathbb{R}^n$ are linear independent.*

**Remarks 1.** *(a) Note that the $n+1$ points $0, e_1, \ldots, e_n \in \mathbb{R}^n$ are affine independent.*

*(b) Affine independence does not depend on the order of the points. Namely, one can check that the vectors $P_1 - P_0, \ldots, P_k - P_0$ are linear independent if and only if the vectors $P_0 - P_i, \ldots, P_{i-1} - P_i, P_{i+1} - P_i, \ldots, P_k - P_i$ are linear independent.*

*(c) Recall that, for given $k+1$ points $P_0, \ldots, P_k$ in $\mathbb{R}^n$ with $n \geq k$, there is always a $k$-dimensional affine plane containing them. These points are affine independent, if they don't lie in an affine plane of dimension $< k$. In particular, three points in $\mathbb{R}^2$ are affine independent, if they don't lie on a common line.*

**Theorem 2.6** (Fundamental Theorem of Affine Geometry)**.** *Every affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ maps $n+1$ affine independent points $P_0, \ldots, P_n$ to $n+1$ affine independent points. Given two ordered sets $P_0, \ldots, P_n$ and $Q_0, \ldots, Q_n$ of affine independent points in $\mathbb{R}^n$, there is a unique affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $f(P_i) = Q_i$.*

*Proof.* Let $P_0, \ldots, P_n \in \mathbb{R}^n$ be $n+1$ affine independent points and $f(x) = Ax+b$ with $A \in GL(n,\mathbb{R})$. Then

$$f(P_1) - f(P_0) = A(P_1 - P_0), \ldots, f(P_n) - f(P_0) = A(P_n - P_0).$$

Since $P_1 - P_0, \ldots, P_n - P_0$ are linear independent and $A \in GL(n,\mathbb{R})$, we conclude that $A(P_1 - P_0), \ldots, A(P_n - P_0)$ are linear independent. This shows that $f(P_0), \ldots, f(P_n)$ are affine independent.

Now, we are given two set $P_0, \dots, P_n$ and $Q_0, \dots, Q_n$ of affine independent points in $\mathbb{R}^n$. We prove existence and uniqueness of an affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(P_i) = Q_i$.

**Existence:** Let $v_i = P_i - P_0$ and $w_i = Q_i - Q_0$. Then $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are both linear independent sets of vectors. Let $A_1 = (v_1 \dots v_n)$ and $A_2 = (w_1 \dots w_n)$. Then $A_1, A_2 \in GL(n, \mathbb{R})$ and $C := A_2 A_1^{-1} \in GL(n, \mathbb{R})$ satisfies $Cv_i = A_2 e_i = w_i$. Thus we have

$$C(P_i - P_0) = Q_i - Q_0 \quad \text{for } i = 1, 2, \dots, n.$$

The affine map $f(x) = Cx + (Q_0 - CP_0)$ satisfies $f(P_0) = Q_0$ and

$$f(P_i) = C(P_i - P_0) + Q_0 = (Q_i - Q_0) + Q_0 = Q_i \quad \text{for } i = 1, 2, \dots, n.$$

**Uniqueness:** Note first that an affine transformation $k(x) = Ax + b$ with $A \in GL(n, \mathbb{R})$ with $k(0) = 0$ and $k(e_i) = e_i$ for $i = 1, \dots, n$ must be the identity map: $k(x) = x$: $k(0) = 0$ implies that $b = 0$, i.e., $k(x) = Ax$ is linear and $k(e_i) = e_i$ implies $k(x) = x$ for all $x \in \mathbb{R}^n$. Now, let $f, g$ be two affine transformations satisfying

$$f(P_i) = g(P_i) = Q_i \quad \text{for } i = 0, 1, \dots, n.$$

Let $h$ be an affine transformation satisfying $h(0) = P_0$ and $h(e_i) = P_i$ for $i = 1, \dots, n$ (existence of such a $h$ is guaranteed by the previous arguments). Then

$$h^{-1} \circ g^{-1} \circ f \circ h(0) = h^{-1}(g^{-1}(f(P_0))) = h^{-1}(g^{-1}(Q_0) = h^{-1}(P_0) = 0$$

and

$$h^{-1} \circ g^{-1} \circ f \circ h(e_i) = h^{-1}(g^{-1}(f(P_i))) = h^{-1}(g^{-1}(Q_i)) = h^{-1}(P_i) = e_i.$$

This shows that $h^{-1} \circ g^{-1} \circ f \circ h = \mathrm{id}_{R^n}$, i.e.

$$f \circ h = g \circ h,$$

and applying $h^{-1}$ from the right on both sides yields

$$f = g.$$

□

## 2.3 Normal forms in conjugation classes

Next, for a given affine transformation $f(x) = Ax + b$, $A \in GL(2, \mathbb{R})$, $b \in \mathbb{R}^n$, we look at all its conjugates $g^{-1} \circ f \circ g$ with $g \in A(\mathbb{R}^2)$ and try to find a particularly simple form. We first deal with the linear part, since we know that

$$(g^{-1} \circ f \circ g)' = (g^{-1})' \circ f' \circ g',$$

where $h'$ denotes the linear part $h'(x) = Cx$ of an affine transformation $h(x) = Cx + d$.

**Proposition 2.7.** *Let $A \in GL(2, \mathbb{R})$. Then there exists a $C \in GL(2, \mathbb{R})$ such that $C^{-1}AC$ is one of the following three* normal forms*:*

$$\begin{aligned}
C^{-1}AC &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}, \\
C^{-1}AC &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R}, \beta \neq 0, \\
C^{-1}AC &= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad \lambda \in \mathbb{R}.
\end{aligned}$$

*Proof.* Let $p(t) = \det(t\mathrm{Id} - A)$ denote the characteristic polynomial of $A$. In $\mathbb{C}$, this polynomial of degree 2 is a product of the form

$$p(t) = (t - \lambda_1)(t - \lambda_2), \qquad \lambda_1, \lambda_2 \in \mathbb{C}.$$

We distinguish three cases:

(a) $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$: Then there are two linear independent eigenvectors $v_1, v_2 \in \mathbb{R}^2$ and we have with $C := (v_1\, v_2) \in GL(2, \mathbb{R})$:

$$C^{-1}AC = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

(b) One of the eigenvalues is not real, i.e., $\lambda_1 = \alpha + i\beta$ with $\beta \neq 0$. Since $p(t)$ is a real polynomial, we have $\overline{p(t)} = p(\bar{t})$, and therefore

$$p(\overline{\lambda_1}) = \overline{p(\lambda_1)} = \overline{0} = 0,$$

i.e., $\lambda_2 = \overline{\lambda_1} = \alpha - i\beta$. Then

$$p(t) = (t - \lambda_1)(t - \overline{\lambda_1}) = t^2 - 2\alpha t + \alpha^2 + \beta^2.$$

Then the theorem of Cayley-Hamilton yields

$$A^2 - 2\alpha A + (\alpha^2 + \beta^2)\mathrm{Id} = 0,$$

which implies that

$$(A - \alpha\mathrm{Id})^2 = -\beta^2\mathrm{Id}.$$

Let $J = \beta{-}1(A - \alpha\mathrm{Id})$. Then

$$J^2 = \beta^{-2}(A - \alpha\mathrm{Id})^2 = -\mathrm{Id},$$

i.e. $J \in GL(2, \mathbb{R}^2)$. Now let $v_1 \neq 0$ be an arbitrary vector and $v_2 = Jv_1$. If $v_1$ and $v_2$ were linear dependent, then we would have $v_2 = \mu v_1$ and $\mu^2 v_1 = J^2 v_1 = -v_1$, a contradiction. Thus $v_1$ and $v_2$ are linear independent and $C = (v_1\, v_2) \in GL(2, \mathbb{R})$. From

$$v_2 = Jv_1 = \beta^{-1}(A - \alpha\mathrm{Id})v_1 = \beta^{-1}(Av_1 - \alpha v_1)$$

we conclude that

$$Av_1 = \alpha v_1 + \beta v_2,$$

and from

$$-v_1 = J^2 v_1 = Jv_2 = \beta^{-1}(A - \alpha\mathrm{Id})v_2 = \beta^{-1}(Av_2 - \alpha v_2)$$

we conclude that

$$Av_2 = -\beta v_1 + \alpha v_2.$$

This implies with $C = (v_1\, v_2)$:

$$C^{-1}AC = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

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(c) We have $\lambda_1 = \lambda_2 \in \mathbb{R}$. Then $p(t) = (t-\lambda)^2 = t^2 - 2\lambda t + \lambda^2$ with $\lambda := \lambda_1$. By Cayley-Hamilton, we also have

$$(A - \lambda \mathrm{Id})^2 = 0.$$

This implies that $A - \lambda\mathrm{Id}$ is not injective, for otherwise it would also be surjective and $A - \lambda\mathrm{id}$ as well as $(A-\lambda\mathrm{Id})^2$ would both be bijective. Let $V := \ker(A - \lambda\mathrm{Id})$. Since $A - \lambda\mathrm{Id}$ is not injective, we have $V \neq \{0\}$. If $V = \mathbb{R}^2$ then $A = \lambda\mathrm{Id}$ and

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Otherwise, choose a vector $v_2 \notin V = \ker(A - \lambda\mathrm{Id})$ and $v_1 = (A - \lambda\mathrm{Id})v_2 \neq 0$. Then $v_1, v_2$ cannot be linear dependent, for otherwise we would have $v_1 = \mu v_2$ with $\mu \neq 0$ and

$$0 = (A - \lambda\mathrm{Id})^2 v_2 = (A - \lambda\mathrm{Id})v_1 = \mu(A - \lambda\mathrm{Id})v_2 = \mu v_1 \neq 0.$$

Moreover, $0 = (A - \lambda\mathrm{Id})^2 v_2 = (A - \lambda\mathrm{Id})v_1$ implies that

$$Av_1 = \lambda v_1,$$

and we also have, by the construction of $v_1$,

$$Av_2 = v_1 + \lambda v_2.$$

Thus, if we choose $C = (v_1\, v_2)$, we obtain

$$C^{-1}AC = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

□

The above proposition is useful for the proof of the following result:

**Theorem 2.8.** *Let $f(x) = Ax + b$ with $A \in GL(2,\mathbb{R})$ and $b \in \mathbb{R}^2$ be an affine transformation. Then $f$ is conjugate to one of the following normal forms:*

$$\begin{aligned}
g(x) &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x, \qquad \lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\}, \lambda_1 \geq \lambda_2, \\
g(x) &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} x, \qquad \alpha, \beta \in \mathbb{R}, \beta \neq 0, \\
g(x) &= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} x, \qquad \lambda \in \mathbb{R}\setminus\{0\}, \\
g(x) &= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}\setminus\{0\}, \\
g(x) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

*Part of proof.* One first chooses $C \in GL(2, \mathbb{R})$ as in Proposition 2.7 and obtains with $h(x) = Cx$ and $b_1 = C^{-1}b$:

$$h^{-1} \circ f \circ h(x) = C^{-1}(A(Cx) + b) = (C^{-1}AC)x + C^{-1}b = (C^{-1}AC)x + b_1.$$

Now, $A_1 := C^{-1}AC$ has one of the forms in Proposition 2.7. Thus $f$ is conjugate to $f_1(x) = A_1x + b_1$. If $\det(I - A_1) \neq 0$, then $I - A_1$ is invertible and $v = (I - A_1)^{-1}b$ satisfies $b = v - A_1v$. Then $f_2 = t_{-v} \circ f_1 \circ t_v$ is conjugate to $f_1$ and

$$f_2(x) = f_1(x + v) - v = A_1(x + v) + b_1 - v = A_1x + b_1 - (v - A_1v) = A_1x,$$

i.e., we are able to remove the translational part of $f_1$ in this case by conjugation with $t_v$. If $\det(I - A_1) = 0$, we cannot completely remove the translational part, but we can simplify the translational part into the form presented in the theorem. Finally, note that the first three normal forms have a fixed point, namely $x = 0$, whereas the last two don't have fixed points. □

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## 2.4 Applications: Ceva's and Menelaus' Theorem

We like to finish this chapter by looking at two classical theorems of affine geometry. Before so doing, let me recall two important formulas for the ratios of three points $P, Q, R$ on a line $l$. If the coordinates are given by $P = (x_p, y_p)$, $Q = (x_q, y_q)$ and $R = (x_R, y_R)$ and $l$ is not parallel to the $y$-axis, then the $x$-coordinate formula tells us that

$$\frac{PQ}{QR} = \frac{x_Q - x_P}{x_R - x_Q}$$

where $\frac{PQ}{QR}$ denotes the ratio of the segments $PQ$ and $QR$ (which can be negative if $Q$ doesn't lie between $P$ and $R$). If $l$ is not parallel to the $x$-axis, then the $y$-coordinate formula tell us that

$$\frac{PQ}{QR} = \frac{y_Q - y_P}{y_R - y_Q}.$$

**Theorem 2.9** (Ceva's Theorem)**.** *Let $\Delta ABC$ be a triangle, and let $X$ be a point which does not lie on any of its extended sides. If the line $l_{AX}$ through $A$ and $X$ meets the line $l_{BC}$ in $P$, $l_{BX}$ meets $l_{CA}$ in $Q$ and $l_{CX}$ meets $l_{BA}$ in $R$, then*

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$

*Proof.* Let $\Delta$ be the standard triangle with the vertices $0, e_1, e_2$. By the Fundamental Theorem of Affine Geometry, there is an affine transformation $f$ which maps $\Delta ABC$ to $\Delta$ and $f(A) = 0$, $f(B) = e_1$ and $f(C) = e_2$. Since affine transformations map straight lines to straight lines and preserve ratios along lines, we only have to prove Ceva's Theorem for the standard triangle. Thus let $A = 0$, $B = e_1$ and $C = e_2$. For a given point $X = (u, v)$, which doesn't lie on the $x$- and $y$-axis and on the line $y = 1 - x$, one easily calculates the intersections

$$\begin{aligned} R = l_{CX} \cap l_{AB} &= \left( \frac{u}{1-v}, 0 \right), \\ P = l_{AX} \cap l_{BC} &= \left( \frac{u}{u+v}, \frac{v}{u+v} \right), \\ Q = l_{BC} \cap l_{CA} &= \left( 0, \frac{v}{1-u} \right). \end{aligned}$$

Here we used the fact that the line through the points $Z = (a, b)$ and $X = (u, v)$ is given by

$$l_{ZX} = \{\lambda(a, b) + (1 - \lambda)(u, v) \mid \lambda \in \mathbb{R}\}.$$

Using the $x$- and $y$-coordinate formulas, we conclude that

$$\begin{aligned} &\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} \\ &\qquad = \frac{u/(1-v)}{1 - u/(1-v)} \cdot \frac{u/(u+v) - 1}{0 - u/(u+v)} \cdot \frac{v/(1-u) - 1}{0 - v/(1-u)} \\ &\qquad\qquad = \frac{u}{1-u-v} \cdot \frac{-v}{-u} \cdot \frac{u+v-1}{-v} = 1. \end{aligned}$$

□

Let us finally present the Theorem of Menelaus. The proof is Exercise 12.

**Theorem 2.10** (Theorem of Menelaus)**.** *Let $\Delta ABC$ be a triangle and let $l$ be a line that crosses the extended sides $l_{BC}$, $l_{CA}$, $l_{AB}$ at the points $P, Q, R$, respectively. Then*

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

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# 3 Projective Geometry

## 3.1 Points, homogeneous coordinates and Lines

*Projective Geometry* was discovered through artists' attempts to capture the three-dimensional world on a two-dimensional screen. In the early Middle Ages, artists started to produce accurate reproductions of three dimensional scenes by using *methods of perspective*. A prominent artist who studied the mathematical background of this problem was Albrecht Dürer (1471-1528) from Germany.

The basic idea is to draw rays from a reference point $p$ (position of the artist's eye) to points $q$ of a three-dimensional object behind a screen $\pi \cong \mathbb{R}^2$. These rays intersect the screen at the image points $f(q) \in \pi$. Obviously, objects far behind the screen have smaller images on the screen as the same objects closer to the screen.

We can think of the map $f$ as a central projection of $\mathbb{R}^3$ with respect to the centre $p$ to the screen $\pi$. From a mathematical point of view, $f$ is also well defined for points between the eye and the screen as well as for points behind the eye. Any point $q \in \mathbb{R}^3$ is mapped to the screen $\pi$ by taking the extended straight line $l_{pq}$ and defining

$$f(q) = l_{p1} \cap \pi.$$

However, the points in the plane $E$ through $p$ parallel to $\pi$ don't have images in $\pi$. So we have

$$f : \mathbb{R}^3 \setminus E, \quad f(q) := l_{pq} \cap \pi.$$

Thus we can identify lines $l$ through $p \in \mathbb{R}^3$ with points on the screen $\pi \cong \mathbb{R}^2$ with the exception of the lines in $E \parallel \pi$ through $p$. One could consider those lines to correspond to points which are infinitely far away on the screen $\pi$ and call them *ideal* points.

The *projective plane* $\mathbb{R}P^2$ is obtained by choosing the reference point $p = 0 \in \mathbb{R}^3$, removing the screen $\pi$ and only looking at all lines through $p$ as *Points* in $\mathbb{R}P^2$. Since Points in $\mathbb{R}P^2$ are actually lines in $\mathbb{R}^3$, we use a capital starting letter 'P' to emphasize this distinction. Hence, the elements of $\mathbb{R}P^2$ are *projetive points* or *Points*. These Points become ordinary points when choosing a screen $\pi$, but by such a choice we will always miss out the Points parallel to this screen, which we refer to as *ideal Points* with respect to the screen $\pi$.

*Klein viewpoint* was that a geometry is given by a group of transformations acting on a space of points (and preserving some geometric properties). We will first have a closer look at the space of points, which we call a projective space and will later introduce the corresponding group of projective transformations.

Let us already now state a significant difference between affine and projective space: In 2-dimensional affine space there is a unique line through two different points, but not every two different lines intersect in a unique point. Two different parallel lines don't have an intersection point. In 2-dimensional projective geometry, any two different lines intersect in a unique point. The notion of parallelity does no longer exist in projective geometry.

Projective spaces can be defined for arbitrary fields $\mathbb{F}$ in any dimension $n \geq 1$. If you don't feel comfortable with arbitrary fields, then simply think of $\mathbb{F}$ as being $\mathbb{R}$ or $\mathbb{C}$:

**Definition 3.1.** *An n-dimensional* projective spcae *over a field $\mathbb{F}$ is the set of all 1-dimensional subspaces of $\mathbb{F}^{n+1}$ and is denoted by $\mathbb{F}P^n$. Any non-zero vector $v \in \mathbb{F}^{n+1}$ determines a Point $[v] := \mathbb{F} \cdot v \in \mathbb{F}P^n$.*

It can be shown that $\mathbb{C}P^1$ is topologically the same as the 2-sphere $S^2$. But in this course, we restrict ourselves mainly to the 2-dimensioal real projective space $\mathbb{R}P^2$. Note that every $v = (v_1, v_2, v_3) \in \mathbb{R}^3 \setminus \{0\}$ defines a straight line

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \mathbb{R} \cdot v \in \mathbb{R}P^2.$$

We call $[v_1, v_2, v_3]$ the *homogeneous coordinates* of the projective point and we have

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \longleftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\} : \, v = \lambda w.$$

**Definition 3.2.** *A* Line $l \subset \mathbb{R}P^2$ *(note the choice of the capital letter 'L' since this is a projective line) is uniquely associated to a plane $E_l \subset \mathbb{R}^3$ through the origin: all Points in $l$ are the lines through the origin which lie in $E_l$.*

Let $l \subset \mathbb{R}P^2$ be a Line. If we choose a screen $\pi \subset \mathbb{R}^3$ which is not parallel to $E_l$, then the Points of $l$ intersecting $\pi$ form a line in $\pi$, namely $\pi \cap E_l$. But $l$ contains one more Point, which has no image in $\pi$, namely the line through the origin in $E_l$ which is parallel to $\pi$. This Point is ideal with respect to the screen $\pi$.

Next, we show that any two different Lines in $\mathbb{R}P^2$ intersect:

**Proposition 3.3.** *Any two different Lines $l_1, l_2 \subset \mathbb{R}P^2$ intersect in a unique Point.*

*Proof.* The corresponding planes $E_{l_1}, E_{l_2} \subset \mathbb{R}^3$ have non-empty intersection (namely the origin). Therefore, they must intersect in a whole line through the origin. This line is the intersection Point of $l_1$ and $l_2$ in $\mathbb{R}P^2$. □

If we choose any screen $\pi$, then all Points of $\mathbb{R}P^2$ have image points in $\pi$, except for the lines through the origin which are parallel to $\pi$. Those lines lie in a plane $E$ parallel to $\pi$ and therefore define a Line in $\mathbb{R}P^2$. We refer to this Line in $\mathbb{R}P^2$ as the *ideal Line* with respect to the screen $\pi$. We can think of $\mathbb{R}P^2$ as the *completion* of $\pi \cong \mathbb{R}^2$ by this ideal Line of Points. This completion process yields the fact that any two different Lines intersect. If these Lines are represented as parallel lines in $\pi$, their intersection Point is ideal with respect to $\pi$, otherwise, their intersection Point is a point in $\pi$. The intersection of the ideal Line with any other (non-ideal) Line $l$ is precisely the ideal Point of $l$.

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**Proposition 3.4.** *Let $[v], [w] \in \mathbb{R}P^2$ be two different Points. Then the Line $l$ through $[v], [w]$ is given by*

$$\left\{ [z] \in \mathbb{R}P^2 \mid \det \begin{pmatrix} v_1 & w_1 & z_1 \\ v_2 & w_2 & z_2 \\ v_3 & w_3 & z_3 \end{pmatrix} = 0 \right\}.$$

*Proof.* $[v], [w]$ are two different Points if $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are two linear independent vectors in $\mathbb{R}^3$. The plane $E_l \subset \mathbb{R}^3$ through the origin associated to the Line $l$ is the span of $v, w$. Any non-vector $z$ in this plane defines a Point $[z]$ on $l$ and vice versa. A non-zero vector $z$ is in the plane $E_l$ if and only if it is a linear combination of $v, w$, which is equivalent to

$$\det \begin{pmatrix} v_1 & w_1 & z_1 \\ v_2 & w_2 & z_2 \\ v_3 & w_3 & z_3 \end{pmatrix} = 0.$$

□

**Example.** *The Line $l$ through $[1, 0, 5]$ and $[2, 6, 0]$ is given by the Points $[z_1, z_2, z_3]$ satisfying*

$$\det \begin{pmatrix} 1 & 2 & z_1 \\ 0 & 6 & z_2 \\ 5 & 0 & z_3 \end{pmatrix} = 0.$$

*This transforms into*

$$6z_3 + 10z_2 - 30z_1 = 0,$$

*or, after division by* 2*:*

$$-15z_1 + 5z_2 + 3z_3 = 0.$$

*Thus,*

$$l = \{[z] \in \mathbb{R}P^2 \mid -15z_1 + 5z_2 + 3z_3 = 0\}.$$

## 3.2 Higher dimensional projective spaces

Let us shortly look at higher dimensional real projective space $\mathbb{R}P^n$.

**Definition 3.5.** *A* $k$-dimensional subspace *of* $\mathbb{R}P^n$ *with* $k \leq n$ *is the set of all one-dimensional subspaces in a* $(k+1)$*-dimensional subspace of* $\mathbb{R}^{n+1}$*.* $(n-1)$*-dimensional subspaces of* $\mathbb{R}P^n$ *are called* Hyperplanes *or* projective hyperplanes.

**Lemma 3.6.** *Let* $E, F \subset \mathbb{R}P^n$ *be two subspaces of dimension* $k$ *and* $l$*, respectively. If* $k + l - n \geq 0$*, then* $E \cap F$ *is a subspace of dimension* $\geq k + l - n$*. If* $E \cap F = \emptyset$*, then there exists a unique subspace of dimension* $k+l+1$*, containing both* $E$ *and* $F$*.*

*Proof.* $E$ and $F$ determine $(k+1)$- and $(l+1)$-dimensional subspaces $\hat{E}, \hat{F}$ of $\mathbb{R}^{n+1}$. By the dimension formula

$$\begin{aligned} \dim \hat{E} \cap \hat{F} &= \dim(\hat{E}) + \dim(\hat{F}) - \dim(\hat{E} + \hat{F}) \\ &\geq (k+1) + (l+1) - (n+1) = (k+l+1) - n. \end{aligned}$$

So, $\hat{E} \cap \hat{F}$ is a subspace of $\mathbb{R}^{n+1}$ of dimension $\geq (k+l+1) - n$ and determines a projective subspace $E \cap F$ of dimension $\geq k + l - n$.

If $E \cap F = \emptyset$, then $\hat{E} \cap \hat{F} = \{0\}$, then

$$\dim \hat{E} + \hat{F} = \dim(\hat{E}) + \dim(\hat{F}) = (k+1) + (l+1) = k + l + 2,$$

and $\hat{E} + \hat{F}$ determines a $(k+l+1)$-dimensional projective space, containing both $E$ and $F$. $\square$

The following corollary is a generalization of Proposition 3.3:

**Corollary 3.7.** *In* $\mathbb{R}P^2$*, any two different Lines intersect in a unique Point. For any two different Points, there exists a unique Line containing both of them. If* $\mathbb{R}P^3$*, any two different Planes intersect in a Line and a Line not lying in a Plane intersects that Plane in a unique Point.*

## 3.3 Two proofs of Desargues' Theorem

Next, we present a highlight, namely *Desargues' Theorem.* We will give two different proofs of this theorem. Beforehand, however, a little bit of notation:

**Definition 3.8.** *Points* $A_1, \ldots, A_n \in \mathbb{R}P^2$ *are* collinear*, if there is a Line containing them. Lines* $l_1, \ldots, l_n \subset \mathbb{R}P^2$ *are* concurrent*, if they contain a common Point, i.e.,* $l_1 \cap \cdots \cap l_n \neq \emptyset$*. Three non-collinear Points* $A, B, C \in \mathbb{R}P^2$ *define a* triangle $\Delta ABC$ *and* $AB, BC, CA$ *its* sides*. Note that the sides of a triangle are not only line segments but the whole projective lines.*

**Theorem 3.9** (Desargues' Theorem)**.** *Let $\Delta P_0P_1P_2$ and $\Delta Q_0Q_1Q_2$ be two triangles in $\mathbb{R}P^2$ such that all three different Lines $P_0Q_0$, $P_1Q_1$ and $P_2Q_2$ meet in a common Point $Z$. Then the three intersection Points*

$$S_{01} = P_0P_1 \cap Q_0Q_1, \quad S_{02} = P_0P_2 \cap Q_0Q_2, \quad S_{12} = P_1P_2 \cap Q_1Q_2$$

*are collinear.*

We present two proofs of this theorem: the first proof is algebraic and the second proof is geometric.

*Algebraic Proof.* We have $P_i \neq Q_i$, for otherwise they would not define a unique Line $P_iQ_i$. Let

$$P_i = [v_i], \quad Q_i = [w_i].$$

Then $v_0, v_1, v_2$ and $w_0, w_1, w_2$ are two sets of linear independent vectors. Let $Z = [u] = P_0Q_0 \cap P_1Q_1 \cap P_2Q_2$. Then

$$0 \neq u = \alpha_0 v_0 + \beta_0 w_0 = \alpha_1 v_1 + \beta_1 w_1 = \alpha_2 v_2 + \beta_2 w_2$$

with $(\alpha_i, \beta_i) \neq 0$. This implies

$$\alpha_0 v_0 - \alpha_1 v_1 = \beta_1 w_1 - \beta_0 w_0 \neq 0,$$

since $(\alpha_i, \beta_i) \neq 0$ and $v_0, v_1$ and $w_0, w_1$ are linear independent. Since $[\alpha_0 v_0 - \alpha_1 v_1] \in P_0P_1$ and $[\beta_1 w_1 - \beta_0 w_0] \in Q_0Q_1$, we have

$$S_{01} = [\alpha_0 v_0 - \alpha_1 v_1].$$

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Similarly, we derive

$$S_{02} = [\alpha_0 v_0 - \alpha_2 v_2], \qquad S_{12} = [\alpha_1 v_1 - \alpha_2 v_2].$$

Note that

$$\det \begin{pmatrix} \alpha_0 v_0 - \alpha_1 v_1 & \alpha_0 v_0 - \alpha_2 v_2 & \alpha_1 v_1 - \alpha_2 v_2 \end{pmatrix} = 0,$$

since the first column of this matrix is equal to the second column minus the third column. But this implies that the three points $S_{01}, S_{02}, S_{12}$ are collinear. □

*Geometric Proof.* We first think of the two triangles $\Delta P_0P_1P_2$ and $\Delta Q_0Q_1Q_2$ lying in two different Planes $F_P$ and $F_Q$ of $\mathbb{R}P^3$. We assume that the Lines $Q_0P_0$, $Q_1P_1$ and $Q_2P_2$ intersect in a Point $Z \in \mathbb{R}P^3$. At the end of this proof, we will explain how to bring all seven Points into one projective plane. Let $E \subset \mathbb{R}P^3$ be the Plane containing the two concurrent Lines $Q_0P_0$ and $Q_1P_1$. By our above assumption $P_2$ and $Q_2$ don't lie in $E$, but in the planes $F_P$ and $F_Q$. Let $l = F_P \cap F_Q$ be the intersection Line.

For $i \neq j$, the five Points $Z, P_i, Q_i, P_j, Q_j$ lie in a common Plane, hence $P_iP_j$ and $Q_iQ_j$ intersect in a Point $S_{ij}$. Since $P_iP_j$ lies in $F_P$ and $Q_iQ_j$ lies in $F_Q$, the intersection Point $S_{ij}$ lies on the Line $l = F_P \cap F_Q$. Hence, all three points $S_{01}, S_{02}$ and $S_{12}$ lie on the same line $l$ and are thus collinear.

Now, assume that the Points $P_2, Q_2 \notin E$ are converging to limit Points inside $E$, which finally yields the 2-dimensional statement by this limiting argument. □

## 3.4 Group of projective transformations

Now, we introduce the transformation group associated with Projective Geometry. A linear map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with

$$f(v) = Av, \qquad A \in GL(n+1, \mathbb{R}),$$

has the property that $f(\lambda v) = \lambda f(v)$, for all $\lambda \in \mathbb{R}$. This implies that $f$ induces a map on $\mathbb{R}P^n$, namely

$$f([v]) := [f(v)].$$

Such a map is called a *projective transformation*.

**Proposition 3.10.** *A map $f : \mathbb{R}P^n \to \mathbb{R}P^n$, given by*

$$f([v]) = [Av],$$

*with $A \in GL(n+1, \mathbb{R})$ is called a* projective transformation. *The set of all projective transformations forms a group under composition, the so-called* group of projective transformations *$P(\mathbb{R}P^n)$.*

*Proof.* Let $f, g : \mathbb{R}P^n \to \mathbb{R}P^n$ be given by

$$f([v]) = [Av], \quad g([v]) = [Bv], \qquad A, B \in GL(n+1, \mathbb{R}).$$

Then we have

$$f \circ g([v]) = f([Bv]) = [(AB)v], \qquad AB \in GL(n+1, \mathbb{R}),$$

and the inverse of $f$ is given by $f^{-1}([v] = [A^{-1}v]$, since

$$f^{-1}(f([v])) = [(A^{-1}A)v] = [v].$$

□

**Remark 4.** *Let $A \in GL(n+1, \mathbb{R})$ and $\lambda \neq 0$. Then $f([v]) = [Av]$ and $g([v] = [(\lambda A)v]$ define the same projective transformation. Therefore, we can identify the group $P(\mathbb{R}P^n)$ canonically with*

$$GL(n+1, \mathbb{R})/\sim,$$

*where $A \sim B$ if there is a $\lambda \neq 0$ such that $A = \lambda B$. We also write*

$$PGL(n+1, \mathbb{R})$$

*for $GL(n+1, \mathbb{R})/\sim$ and call this group the* projective general linear group.

In order to state our next result on projective transformations, we first have to introduce the notion of *points in general position*:

**Definition 3.11.** *$n+2$ Points in $\mathbb{R}P^n$ are called* in general position *if no $n+1$ Points of them lie in a projective hyperplane.*

**Example.** *The Points*

$$p_0 = [1,0,0],\ p_1 = [0,1,0],\ p_2 = [0,0,1],\ p_3 = [1,1,1] \in \mathbb{R}P^2$$

*are* in general position*, whereas the Points*

$$p_0,\ p_1,\ p_2,\ q_3 = [0,1,1] \in \mathbb{R}P^2$$

*are not, since $p_1, p_2, q_3$ lie on a common Line.*

**Theorem 3.12** (Fundamental Theorem of Projective Geometry)**.** *Let $p_0, \dots, p_{n+1}$ and $q_0, \dots, q_{n+1}$ be two sets of Points* in general position *in $\mathbb{R}P^n$. Then there exists a unique projective transformation $f : \mathbb{R}P^n \to \mathbb{R}P^n$ such that*

$$f(p_i) = q_i \quad \textit{for } i = 0, 1, \dots, n+1.$$

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*Proof.* Let $p_i = [v_i]$ and $q_i = [w_i]$. $v_0, \dots, v_n$ and $w_0, \dots, w_n$ are both bases of $\mathbb{R}^{n+1}$, since both sets of points are in general position. This implies that we can express $v_{n+1}$ and $w_{n+1}$ as linear combinations

$$v_{n+1} = \sum_{i=0}^{n} \alpha_i v_i, \qquad w_{n+1} = \sum_{i=0}^{n} \beta_i w_i.$$

Note that general position implies that all $\alpha_i \neq 0$ and also all $\beta_i \neq 0$. (If there were $i$ with $\alpha_i = 0$, then $v_0, \dots, v_{i_1}, v_{i+1}, \dots, v_{n+1}$ were linear dependent, contradicting to the assumption of general position.)

Define $v_i' = \alpha_i v_i$ for $i = 0, \dots, n$ and $v_{n+1}' = v_{n+1}$ and $w_i' = \beta_i w_i$ for $i = 0, \dots, n$ and $w_{n+1}' = w_{n+1}$. This implies that $v_0', \dots, v_n'$ and $w_0', \dots, w_n$@ are both bases of $\mathbb{R}^{n+1}$ and

$$v_{n+1}' = \sum_{i=0}^{n} v_i', \quad w_{n+1}' = \sum_{i=0}^{n} w_i'.$$

Now, choose a matrix $A \in GL(n+1, \mathbb{R})$ such that $Av_i' = w_i'$ for $i = 0, \dots, n$. This implies that we also have

$$Av_{n=1}' = A(v_0' + \dots + v_n') = w_0' + \dots + w_n' = w_{n+1}',$$

and therefore the projective transformation $f([v]) = [Av]$ satisfies

$$f(p_i) = q_i \quad \text{for } i = 0, 1, \dots, n+1.$$

Let us, finally, prove uniqueness: Let $g[v] = [Bv]$ satisfy $g(p_i) = q_i$ for $i = 0, \dots, n+1$. Then

$$Bv_i' = \lambda_i w_i' \quad \text{with } \lambda_i \neq 0.$$

We have to show that $\lambda_0 = \dots = \lambda_{n+1} = \lambda$, since then $B = \lambda A$ and $g = f$. Now,

$$\begin{aligned} Bv_{n+1}' &= \lambda_{n+1} w_{n+1}' = \lambda_{n+1} w_0' + \dots + \lambda_{n+1} w_n' \\ &= B(v_0' + \dots + v_n') \\ &= Bv_0' + \dots + Bv_n' \\ &= \lambda_0 w_0' + \dots + \lambda_n w_n'. \end{aligned}$$

Since $w_0', \dots, w_n'$ is a basis of $\mathbb{R}^{n+1}$, we conclude from this that $\lambda_0 = \dots, = \lambda_n = \lambda_{n+1}$. ∎

## 3.5 Duality

For simplicity, we only discuss duality for $\mathbb{R}P^2$.

Duality is a principle which translates every true statement about relations between Points and Lines in $\mathbb{R}P^2$ into a dual statement, which is automatically also true.

We use the map "linear subspace of $\mathbb{R}^3 \mapsto$ linear subspace of $\mathbb{R}^3$", given by

$$U \mapsto U^\perp.$$

This map has the following properties:

(a) $\dim U^\perp = 3 - \dim U$,

(b) $(U^\perp)^\perp = U$,

(c) $(U+V)^\perp = U^\perp \cap V^\perp$,

(d) $(U \cap V)^\perp = U^\perp + V^\perp$.

Note that every linear subspace $U \subset \mathbb{R}^3$ with $\dim U = 1$ defines a unique Point in $\mathbb{R}P^2$ and with $\dim U = 2$ defines a unique Line in $\mathbb{R}P^2$. The *duality principle* is not the following (which we don't prove):

**Theorem 3.13** (Principle of Duality). *A statement about finitely many Lines and Points, inclusions, intersections and joinings remains true if we perform the following replacements:*

$$\begin{array}{rcl} \textit{projective line} & \leftrightarrow & \textit{projective point} \\ \textit{inclusion} & \leftrightarrow & \textit{containment} \\ \textit{intersection} & \leftrightarrow & \textit{joining} \end{array}$$

**Example.** *Three non-collinear Points $p_0, p_1, p_2$ determine a triangle in $\mathbb{R}P^2$. Their dual objects are three non-concurrent lines $l_0, l_1, l_2$. They intersect again in three Points $Q_0 = l_1 \cap l_2$, $Q_1 = l_0 \cap l_2$ and $Q_2 = l_0 \cap l_1$ which forms a dual triangle. The dual statement to Desargues' Theorem reads as follows:*

Desargues: *Given 2 triangles $\Delta P_0P_1P_2$ and $\Delta Q_0Q_1Q_2$. Assume that $P_0Q_0$, $P_1Q_1$ and $P_2Q_2$ are concurrent.*

*Then the intersection Points $P_iQ_i \cap P_jQ_j$ are collinear.*

Dual statement: *Given two pairs of non-concurrent lines $l_0, l_1, l_2$ and $l_0', l_1', l_2'$. Assume that $l_0 \cap l_0'$, $l_1 \cap l_1'$ and $l_2 \cap l_2'$ are collinear.*

*Then the lines joining $l_i \cap l_i'$ and $l_j \cap l_j'$ are concurrent.*

*One realizes that the dual statement to Desargues is an exchange of the assumption and the conclusion of Desargues.*

## 3.6 Cross Ratios

Now we investigate which properties are preserved under projective transformations. According to Klein's viewpoint, these properties are called *properties of projective geometry.* Recall that properties of affine geometry are

(a) being a straight line

(b) parallelity of two lines

(c) ratios of lengths along a straight line

Property (a), is of course, also a property of projective geometry, whereas properties (b) and (c) are not. Other obvious properties of projective geometry are *collinearity* and *concurrency*. We will show that the *cross ratio* of four Points on a Line is also preserved under projective transformations and is therefore another property of projective geometry.

**Definition 3.14.** *Let $p_0, p_1, p_2, p_3 \in \mathbb{R}P^2$ be four different collinear Points with homogeneous coordinates $p_i = [v_i]$, $v_i \in \mathbb{R}^3 \backslash \{0\}$. Assume that*

$$v_2 = \alpha v_0 + \beta v_1 \quad \text{and } v_3 = \gamma v_0 + \delta v_1.$$

*The* cross-ratio $[p_0, p_1; p_2, p_3]$ *is then defined as*

$$[p_0, p_1; p_2, p_3] = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma}. \tag{3}$$

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**Remarks 2.** *(a) One easily sees from the definition that the cross-ratio of four different collinear Points can never be equals* 0 *and* 1.

*(b) For the cross-ratio to be* **well-defined***, we have to show that the expression* (3) *is independent of the choice of homogeneous coordinates. If $p_i = [w_i]$ are different homogeneous coordinates, then we have $w_i = \lambda_i v_i$ with $\lambda_i \neq 0$ and*

$$\begin{aligned} w_2 &= \lambda_2 v_2 = \lambda_2 \alpha v_0 + \lambda_2 \beta v_1 = \frac{\lambda_2}{\lambda_0} \alpha w_0 + \frac{\lambda_2}{\lambda_1} \beta w_1, \\ w_3 &= \lambda_3 v_3 = \lambda_3 \gamma v_0 + \lambda_3 \delta v_1 = \frac{\lambda_3}{\lambda_0} \gamma w_0 + \frac{\lambda_3}{\lambda_1} \delta w_1. \end{aligned}$$

*With these coefficients, the cross-ratio is given as*

$$\frac{\frac{\lambda_2}{\lambda_1}\beta}{\frac{\lambda_2}{\lambda_0}\alpha} \Big/ \frac{\frac{\lambda_3}{\lambda_1}\delta}{\frac{\lambda_3}{\lambda_0}\gamma} = \frac{\lambda_0}{\lambda_1}\frac{\beta}{\alpha} \Big/ \frac{\lambda_0}{\lambda_1}\frac{\delta}{\gamma} = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma},$$

*which proves that cross-ratios are well-defined.*

The following theorem is almost trivial, but it shows that the cross-ratio is an invariant of projective geometry.

**Theorem 3.15.** *Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a projective transformation. Then we have for any four different collinear Points $p_0, p_1, p_2, p_3$:*

$$[f(p_0), f(p_1); f(p_2), f(p_3)] = [p_0, p_1; p_2, p_3],$$

*i.e., cross-ratios are preserved under projective transformations.*

*Proof.* Let $p_i = [v_i]$ and $f([v]) = Av$ with $A \in GL(3, \mathbb{R})$. Then $f(p_i) = [Av_i]$ and if

$$v_2 = \alpha v_0 + \beta v_1 \quad \text{and } v_3 = \gamma v_0 + \delta v_1,$$

then

$$Av_2 = \alpha Av_0 + \beta Av_1 \quad \text{and } Av_3 = \gamma Av_0 + \delta Av_1.$$

Therefore, we have

$$[f(p_0), f(p_1); f(p_2), f(p_3)] = \frac{\alpha}{\beta} \Big/ \frac{\delta}{\gamma} = [p_0, p_1; p_2, p_3].$$

□

**Proposition 3.16.** *Let $[p_0, p_1; p_2, p_3] = x \in \mathbb{R} - \{0, 1\}$. Then, exchanging two of the four entries, we have*

$$[p_1, p_0; p_2, p_3] = [p_0, p_1; p_3, p_2] = \frac{1}{x} \tag{4}$$

*and*

$$[p_0, p_2; p_1, p_3] = [p_3, p_1; p_2, p_0] = 1 - x. \tag{5}$$

*Proof.* The equation (4) is straightforward. We only show that

$$[p_0, p_2; p_1, p_3] = 1 - x.$$

The second formula in (5) is similar. Let $p_i = [v_i]$ and

$$v_2 = \alpha v_0 + \beta v_1 \quad \text{and } v_3 = \gamma v_0 + \delta v_1.$$

This implies that

$$\begin{aligned} v_1 &= -\frac{\alpha}{\beta} v_0 + \frac{1}{\beta} v_2, \\ v_3 &= \gamma v_0 + \delta v_1 = \gamma v_0 + \delta \left( -\frac{\alpha}{\beta} v_0 + \frac{1}{\beta} v_2 \right) \\ &= \frac{\gamma\beta - \alpha\delta}{\beta} v_0 + \frac{\delta}{\beta} v_2. \end{aligned}$$

Thus, the cross-ratio $[p_0, p_2; p_1, p_3]$ is

$$\frac{\frac{1}{\beta}}{-\frac{\alpha}{\beta}} \Big/ \frac{\frac{\delta}{\beta}}{\frac{\gamma\beta - \alpha\delta}{\beta}} = -\frac{1}{\alpha} \Big/ \frac{\delta}{\gamma\beta - \alpha\delta} = \frac{\alpha\delta - \beta\gamma}{\alpha\delta} = 1 - \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma} = 1 - x.$$

□

Next, we state some important results concerning cross-ratios:

The first result states that the cross-ratios of two sets of four Points in perspective coincide:

**Theorem 3.17.** *Let $P_0, P_1, P_2, P_3$ be four different Points on a Line $l_P$ and $Q_0, Q_1, Q_2, Q_3$ be four different Points on a different Line $l_Q$. Assume that the four Lines $P_iQ_i$ for $i = 0, 1, 2, 3$ all meet in a common Point $Z$. Then we have*

$$[P_0, P_1; P_2, P_3] = [Q_0, Q_1; Q_2, Q_3].$$

*Proof.* Note that the Points $P_0, P_1, Q_0, Q_1$ are in general position. Therefore there is a unique projective transformation $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ with $f(P_0) = Q_0$, $f(Q_0) = P_0$, $f(P_1) = Q_1$ and $f(Q_1) = P_1$. Since $f^2$ fixes the four Points $P_0, P_1, Q_0, Q_1$, we must have $f^2 = \mathrm{id}_{\mathbb{R}P^2}$. Moreover, since $f$ maps $l_P$ to $l_Q$ and vice versa, $f$ must fix the intersection point $I = l_P \cap l_Q$. Since $f$ fixes the Lines $P_0Q_0$ and $P_1Q_1$ (as sets), $f$ must also fix the Point $Z$.

We show that $f(P_i) = Q_i$ for $i = 2, 3$: Assume that $f(P_2) = X$ with $X \neq Q_2$. Note that $X \in l_Q$, since $P_2 \in l_P$ and $f$ maps $l_P$ to $l_Q$. Moreover, $X \neq I$. Note also that $X \neq Q_2$ implies that $P_2X$ does not contain $Z$. Therefore, $P_2X$ intersects the two lines $P_0Q_0$ and $P_1Q_1$ in two different Points $R, S \neq Z$. Since $f^2 = \mathrm{id}$, $f$ fixes the Lines $P_2X$, $P_0Q_0$ and $P_1Q_1$ (as sets), and therefore, fixes their intersection Points $R, S$. Note that $Z \notin RS$, since $P_2X \neq P_2Q_2$. Thus $f$ fixes the four Points $R, S, Z, I$, which are in general position. Therefore, we must have $f = \mathrm{id}$, which is a contradiction to $f(P_0) = Q_0$ and the fact that $P_0 \neq Q_0$. Therefore, we conclude that $f(P_2) = Q_2$.

Similarly, we prove $f(P_3) = Q_3$ and conclude with Theorem 3.15 that

$$[Q_0, Q_1; Q_2, Q_3] = [f(P_0), f(P_1); f(P_2), f(P_3)] = [P_0, P_1; Q_0, Q_1].$$

□

**Theorem 3.18.** *Let $P_0, P_1, P_2, P_3$ and $P_0, Q_1, Q_2, Q_3$ be two sets of different collinear Points (on different Lines $l_P$ and $l_Q$ through $P_0$) such that $[P_0, P_1; P_2, P_3] = [P_0, Q_1; Q_2, Q_3]$. Then the Lines $P_1Q_1$, $P_2Q_2$ and $P_3Q_3$ are concurrent.*

*Proof.* Let $Z$ be the intersection Point of $P_1Q_1$ and $P_2Q_2$. Let $X = l_Q \cap P_3Z$. We have to show that $X = Q_3$. Since $P_0, P_1, P_2, P_3$ and $P_0, Q_1, Q_2, X$ are in perspective, we conclude from Theorem 3.17 and the assumption of the theorem that

$$[P_0, Q_1; Q_2, Q_3] = [P_0, P_1; P_2, P_3] = [P_0, Q_1; Q_2, X].$$

Let $P_0 = [v_0], Q_1 = [v_1], Q_2 = [v_2]$, $Q_3 = [v_3]$ and $X = [v_3']$. Assume that

$$\begin{aligned} v_2 &= \alpha v_0 + \beta v_1, \\ v_3 &= \gamma v_0 + \delta v_1, \\ v_3' &= \gamma' v_0 + \delta' v_1. \end{aligned}$$

Since the cross-ratios agree, we conclude that

$$\frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma} = \frac{\beta}{\alpha} \Big/ \frac{\delta'}{\gamma'},$$

which implies that there is $\lambda \neq 0$ such that $\gamma' = \lambda\gamma$ and $\delta' = \lambda\delta$. Thus $v_3' = \lambda v_3$ and $X = [v_3'] = [v_3] = Q_2$. □

Finally, we use Theorems 3.17 and 3.18 to prove the following theorem:

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**Theorem 3.19** (Pappus' Theorem)**.** *Let $P_0, P_1, P_2$ be three Points on a Line $l_P$ and $Q_0, Q_1, Q_2$ be three Points on a different Line $l_Q$. For $0 \leq i < j \leq 2$, let $S_{ij} = P_iQ_j \cap Q_iP_j$. Then the three Points $S_{01}, S_{02}$ and $S_{12}$ are collinear.*

*Proof.* Let $I = l_P \cap l_Q$. We also introduce the Points $U = Q_0P_1 \cap P_0Q_2$ and $V = Q_0P_2 \cap P_1Q_2$. Let $l_U$ denote the Line $Q_0U$ and $l_V$ denote the Line $Q_2V$. Then the Points $I, Q_0, Q_1, Q_2$ on $l_Q$ are in perspective from $P_0$ with the Points $P_1, Q_0, S_{01}, U$ on the Line $l_U$. Thus, by Theorem 3.17,

$$[I, Q_0; Q_1, Q_2] = [P_1, Q_0; S_{01}, U].$$

The Points $I, Q_0, Q_1, Q_2$ on $l_Q$ are in perspective from $P_2$ with the Points $P_1, V, S_{12}, Q_2$ on the Line $l_V$. Again, by Theorem 3.17, we have

$$[I, Q_0; Q_1, Q_2] = [P_1, V; S_{12}, Q_2].$$

Both equations imply

$$[P_1, Q_0; S_{01}, U] = [P_1, V; S_{12}, Q_2],$$

where the four Points in the left cross-ratio lie on the Line $l_U$ and the four Points on the right cross-ratio lie on the Line $l_V$. By Theorem 3.18, we conclude that the Lines $Q_0V$, $S_{01}S_{12}$ and $UQ_2$ are concurrent. This implies that the intersection Point $Q_0V \cap UQ_2 = Q_0P_2 \cap P_0Q_2 = S_{02}$ lies on the Line $S_{01}S_{12}$. □

**Remark 5.** *Assume that four different collinear Points $A, B, C, D$ lie on a screen $\pi \subset \mathbb{R}^3$. Thus, they lie also on a straight line in this screen. We state* without proof *that the cross-ratio can also be calculated as*

$$[A, B; C, D] = \frac{AC}{CB} \Big/ \frac{AD}{DB},$$

*where $\frac{XY}{UV}$ denotes the ratio of the segments $XY$ and $UV$ introduced in Section 2.4 (and can be negative!).*

### 3.7 Conics

Recall that a Line in $\mathbb{R}P^2$ is given by a homogeneous equation of degree one, i.e.,

$$l = \{[x_1, x_2, x_3] \in \mathbb{R}P^2 \mid ax_1 + bx_2 + cx_3 = 0\}$$

with $(a, b, c) \in \mathbb{R}^3 \backslash \{0\}$. Homogeneous equations of **degree two** define conics:

**Definition 3.20.** *A* conic *$C \subset \mathbb{R}P^2$ is given by*

$$C = \{[x_1, x_2, x_3] \in \mathbb{R}P^2 \mid q(x_1, x_2, x_3) = 0\},$$

*where $q$ is a nontrivial homogeneous polynomial of degree two, i.e., of the form*

$$q(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_1x_3 + 2fx_2x_3,$$

*with $(a, b, c, d, e, f) \neq 0$. Introducing the symmetric matrix*

$$A := \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix},$$

*we can write*

$$C = \{[x] \in \mathbb{R}P^2 \mid x^\top Ax = 0\}.$$

*We call $C$ a* non-singular conic *if* $\det A \neq 0$.

**Example.** *The name conic stems from the fact that the intersection of*

$$\{x \in \mathbb{R}^3 \backslash \{0\} \mid x^\top Ax = 0\}$$

*with a screen $\pi \subset \mathbb{R}^3$ (an affine Euclidean plane not passing through the origin) is a* conic section. *E.g., if we intersect*

$$\widetilde{C}_1 := \{x \in \mathbb{R}^3 \backslash \{0\} \mid x_1^2 + x_2^2 - x_3^2 = 0\}$$

*with the affine plane $\pi = \{x \in \mathbb{R}^3 \mid x_3 = 1\}$, we obtain a circle*

$$\widetilde{C}_1 \cap \pi = \{(x_1, x_2, 1) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}.$$

*If we intersect*

$$\widetilde{C}_2 := \{x \in \mathbb{R}^3 \backslash \{0\} \mid x_3^2 - x_1 x_2 = 0\}$$

*with $\pi$, we obtain the hyperbola*

$$\widetilde{C}_2 \cap \pi = \{(x_1, x_2, 1) \in \mathbb{R}^3 \mid x_2 = \frac{1}{x_1}\}.$$

*If we intersect*

$$\widetilde{C}_3 := \{x \in \mathbb{R}^3 \backslash \{0\} \mid x_1^2 - x_2 x_3 = 0\}$$

*with $\pi$, we obtain the parabola*

$$\widetilde{C}_3 \cap \pi = \{(x_1, x_2, 1) \in \mathbb{R}^3 \mid x_2 = x_1^2\}.$$

Let $C := \{[x] \in \mathbb{R}P^2 \mid x^\top A x = 0\}$ be a conic and $f : \mathbb{R}P^2 \to \mathbb{R}P^2$, $f([x]) = [Bx]$ with $B \in GL(3, \mathbb{R})$ a projective transformation. The preimage $f^{-1}(C) \subset \mathbb{R}P^2$ is then given by

$$\begin{aligned} f(C) &= \{[B^{-1}x] \in \mathbb{R}P^2 \mid x^\top A x = 0\} \\ &= \{[y] \in \mathbb{R}P^2 \mid y^\top (B^\top A B) y = 0\}. \end{aligned}$$

Now, we know from Linear Algebra that we can find a suitable $B \in GL(3, \mathbb{R})$ such that $\widetilde{A} := B^\top A B$ is one of the following normal forms:

$$\begin{aligned}
\widetilde{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y_1^2 = 0 \ (C \text{ a Line}), \\
\widetilde{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y_1^2 + y_2^2 = 0 \ (C \text{ a Point}), \\
\widetilde{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (y_1 - y_2)(y_1 + y_2) = 0 \ (C \text{ union of two Lines}), \\
\widetilde{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y_1^2 + y_2^2 + y_3^2 = 0 \ (C = \emptyset), \\
\widetilde{A} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad y_1^2 + y_2^2 - y_3^2 = 0 \ (C \neq \emptyset,\ C \text{ non-singular}).
\end{aligned}$$

The first three normal forms are singular, the fourth is empty, so there is only one type of non-empty non-singular conic modulo projective transformations. Henceforth, we only consider non-empty non-singular conics.

Next, we want to introduce tangent lines, polar lines and poles of non-empty non-singular conics.

Let

$$\widetilde{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \backslash \{0\} \mid q(x_1, x_2, x_3) = 0\} = q^{-1}(0) \backslash \{0\}.$$

At a point $(x_1, x_2, x_3) \in \widetilde{C}$, $n = \operatorname{grad} q(x_1, x_2, x_3)$ is normal to $\widetilde{C}$. So the tangent plane of $\widetilde{C}$ at this point is given by

$$n^\perp = \{v \in \mathbb{R}^3 \mid \langle \operatorname{grad} q(x), v\rangle = 0\} = \{v \in \mathbb{R}^3 \mid \frac{d}{ds}|_{s=0} q(x + tv) = 0\}.$$

Since $q(x) = x^\top Ax$, this translates into

$$0 = \frac{d}{ds}|_{s=0}(x + tv)^\top A(x + tv) = v^\top Ax + x^\top At = 2x^\top Av,$$

since $A$ is a symmetric matrix. This motivates the following definition:

**Definition 3.21.** *Let $C \subset \mathbb{R}P^2$ be a non-empty non-singular conic and $[x] \in C$. The* tangent Line *to $C$ at $[x]$ is given by*

$$\{[v] \in \mathbb{R}P^2 \mid x^\top Av = 0\}.$$

*More generally, if $[x] \in \mathbb{R}P^2$ is an arbitrary Point, the* polar Line *of $[x]$ with respect to $C$ is given by*

$$\{[v] \in \mathbb{R}P^2 \mid x^\top Av = 0\}.$$

*Conversely, if $l \subset \mathbb{R}P^2$ is an arbitrary Line, then the* pole *of $l$ with respect to $C$ is the Point $[x] \in \mathbb{R}P^2$ determined by $x^\top Av = 0$ for all $[v] \in l$.*

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## 3.8 The Theorem of Pascal

This subsection is devoted to a beautiful result for non-singular conics, Pascal's Theorem. We start with the following lemma:

**Lemma 3.22.** *A non-singular conic $C \subset \mathbb{R}P^2$ cannot contain a whole projective line $l$.*

*Proof.* Applying a projective transformation we can assume that

$$l = \{[x_1, x_2, x_3] : x_3 = 0\}.$$

Let $C = \{[x] : x^\top Ax = 0\}$ with

$$A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

Then $l \subset C$ would imply that

$$0 = (x_1\, x_2\, 0)A \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = ax_1^2 + bx_2^2 + 2dx_1x_2 \qquad \text{for all } (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\}.$$

But this would mean that $a = b = d = 0$ in contradiction to the assumption that $C$ is non-singular. $\square$

**Remark 6.** *Here we implicitely used the fact that the image of a non-singular conic under a projective transformation is, again, a non-singular conic.*

Recall the following definitions: Let $C \subset \mathbb{R}P^2$ be a non-empty non-singular conic, defined by

$$C = \{[x] \in \mathbb{R}P^2 \mid x^\top A x = 0\}$$

with $A \in GL(3, \mathbb{R})$. Let $[x] \in \mathbb{R}P^2$ and $l \subset \mathbb{R}P^2$ a Point and a Line satisfying

$$x^\top A v = 0 \qquad \text{for all } [v] \in l.$$

Then $[x]$ is called the (unique) *pole* of $l$ and $l$ is called the (unique) *polar Line* of $[x]$ with respect to $C$. In particular, if $[x] \in C$, then $l$ is called the (unique) *tangent Line* to $C$ at $[x]$.

We have the following facts:

**Lemma 3.23.** *Let $l \subset \mathbb{R}P^2$ be a Line and $C \subset \mathbb{R}P^2$ a non-singular conic. Then*

*(a) $l \cap C$ consists of at most two Points.*

*(b) $l \cap C$ is a single Point if and only if $l$ is tangent to $C$.*

*(c) Assume that $l \cap C = \{P, Q\}$ and $l_P$ and $l_Q$ are the tangents to $C$ at $P, Q$. Then $R = l_P \cap l_Q$ is the pole of $l$.*

*Proof.* ad (a): Choose four Points $P_0, P_1, P_2, P_3 \in \mathbb{R}P^2$ in general position such that $P_0, P_1 \in l$ and $P_1 \notin C$ (this is possible because of Lemma 3.22). Apply the projective transformation $P_0 \mapsto [1, 0, 0]$, $P_1 \mapsto [0, 1, 0]$, $P_2 \mapsto [0, 0, 1]$ and $P_3 \mapsto [1, 1, 1]$. By this we can assume that $l = \{[x_1, x_2, x_3] \mid x_3 = 0\}$ and $[0, 1, 0] \notin C$. Any $P = [x_1, x_2, x_3] \in l \cap C$ satisfies then $P = [x_1, x_2, 0]$, and therefore

$$a x_1^2 + b x_2^2 + 2 d x_1 x_2 = 0, \quad x_1 \neq 0,$$

i.e.,

$$b \left( \frac{x_2}{x_1} \right)^2 + 2d \frac{x_2}{x_1} + a = 0,$$

which (as a quadratic equation in $x_2/x_1$ has at most two solutions for $x_1/x_1$, since the left side cannot be identically zero because $C$ is non-singular.

ad (b): Assume first that $l \cap C$ is a single Point, i.e., $l \cap C = \{[x]\}$. Choose $[y] \in l$, $[y] \neq [x]$. Then we have, for each $\lambda \in \mathbb{R}$,

$$[\lambda x + y] \in l$$

and $[\lambda x + y] \notin C$, since $[\lambda x + y] \neq [x]$, which means

$$\begin{aligned} 0 \neq (\lambda x + y)^\top A (\lambda x + y) &= \lambda^2 \underbrace{x^\top A x}_{=0} + 2\lambda (x^\top A y) + y^\top A y \\ &= 2\lambda (x^\top A y) + y^\top A y \qquad \text{for all } \lambda \in \mathbb{R}. \end{aligned}$$

This implies that we must have

$$x^\top A y = 0,$$

i.e., $[y]$ lies on the tangent Line of $C$ at $[x]$. This shows that $l$ coincides with this tangent Line.

Conversely, let $l$ be the tangent Line of $C$ at $[x]$. Assume that $[y] \in l \cap C$ is a Point different to $[x]$. Our goal is derive a contradiction. Every Point of $l$ can now be written as $[\lambda x + \mu y]$, and we have

$$(\lambda x + \mu y)^\top A(\lambda x + \mu y) = \lambda^2 x^\top Ax + 2\lambda\mu x^\top Ay + \mu^2 y^\top Ay.$$

Now, $x^\top Ax = 0$ since $[x] \in C$, $x^\top Ay = 0$, since $[y]$ lies in the tangent Line of $C$ at $[x]$, and $y^\top Ay = 0$, since $[y] \in C$. This would mean that

$$(\lambda x + \mu y)^\top A(\lambda x + \mu y) = 0,$$

i.e., the whole Line $l$ would be contained in $C$. This, however, contradicts to Lemma 3.22.

ad (c): We set $P = [x]$ and $Q = [y]$. Since the Line $l$ contains $P, Q$, we have

$$l = \{[\lambda x + \mu y] \mid (\lambda, \mu) \neq 0\}.$$

Note that $P, Q \in C$ implies

$$x^\top Ax = y^\top Ay = 0.$$

The tangent Lines $l_P$ and $l_Q$ are given by

$$\begin{aligned} l_P &= \{[z] \mid x^\top Az = 0\}, \\ l_Q &= \{[z] \mid y^\top Az = 0\}. \end{aligned}$$

Therefore, the intersection Point $R = [z] = l_P \cap l_Q$ satisfies

$$x^\top Az = y^\top Az = 0.$$

But this implies that

$$(\lambda x + \mu y)^\top Az = 0 \qquad \text{for all } (\lambda, \mu) \neq 0.$$

Taking transposition, we obtain

$$z^\top A(\lambda x + \mu y) = 0 \qquad \text{for all } (\lambda, \mu) \neq 0,$$

i.e.,

$$z^\top Av = 0 \qquad \text{for all } [v] \in l.$$

This means precisely that $R = [z]$ is the pole of $l$. □

**Lemma 3.24.** *Let $C \subset \mathbb{R}P^2$ be a non-singular conic and $P_1, P_2, P_3 \in C$ be three distinct Points. Let $P_4$ be the intersection Point of the tangents at $P_1$ and $P_2$. Then $P_1, P_2, P_3, P_4$ are in general position and applying the projective transformation*

$$P_1 \mapsto [1, 0, 0],\ P_4 \mapsto [0, 1, 0],\ P_2 \mapsto [0, 0, 1],\ P_3 \mapsto [1, 1, 1],$$

*the equation for $C$ transforms into*

$$x_2^2 - x_1 x_3.$$

*Proof.* We first check that $P_1, P_2, P_3, P_4$ are in general position:

- $P_1, P_2, P_3$ cannot be collinear because of Lemma 3.23(a).
- $P_1, P_2, P_4$ cannot be collinear because otherwise $P_1P_2$ would be a tangent at $P_1$ with more than one intersection Point with $C$, contradicting to Lemma 3.23(b).
- $P_1, P_3, P_4$ cannot be collinear because otherwise $P_1P_4$ would be a tangent at $P_1$ with more than one intersection Point with $C$, contradicting to Lemma 3.23(b).
- $P_2, P_3, P_4$ cannot be collinear because otherwise $P_2P_4$ would be a tangent at $P_1$ with more than one intersection Point with $C$, contradicting to Lemma 3.23(b).

This implies that there is a projective transformation with

$$P_1 \mapsto [1,0,0],\ P_4 \mapsto [0,1,0],\ P_2 \mapsto [0,0,1],\ P_3 \mapsto [1,1,1].$$

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Applying this transformation to $C$, we conclude for the corresponding matrix

$$C = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

that

(a) $[1,0,0] \in C \Leftrightarrow (1\,0\,0)A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \Leftrightarrow a = 0,$

(b) $[0,0,1] \in C \Leftrightarrow c = 0,$

(c) $[0,1,0]$ in tangent of $C$ at $[1,0,0]$:

$$(1\,0\,0)A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \Leftrightarrow d = 0,$$

(d) $[0,1,0]$ in tangent of $C$ at $[0,0,1]$:

$$(1\,0\,0)A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \Leftrightarrow f = 0,$$

(e) $[1,1,1] \in C \Leftrightarrow (1\,1\,1) \begin{pmatrix} 0 & 0 & e \\ 0 & b & 0 \\ e & 0 & 0 \end{pmatrix} = 0 \Leftrightarrow 2e + b = 0.$

This implis that we have

$$A \in \mathbb{R} \cdot \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 0 \end{pmatrix},$$

which means that

$$C = \{[x_1, x_2, x_3] \in \mathbb{R}P^2 \mid x_2^2 - x_1x_3 = 0\}.$$

$\square$

Finally, we can state the Theorem of Pascal:

**Theorem 3.25** (Pascal's Theorem)**.** *Let $C \subset \mathbb{R}P^2$ be a non-singular conic and $P_1, P_2, P_3$ and $Q_1, Q_2, Q_3$ six distinct Points on $C$. Let*

$$R_1 = P_2Q_3 \cap P_3Q_2, \quad R_2 = P_1Q_3 \cap P_3Q_1, \quad R_3 = P_1Q_2 \cap Q_1P_2.$$

*Then $R_1, R_2$ and $R_3$ lie on a common Line.*

**Remark 7.** *Note that Pappus' Theorem and Pascal's Theorem are closely related. While Pascal's Theorem is concerned with a non-singular conic, Pappus' Theorem is an analogous statement in the singular case, i.e., when the conic consists of two different Lines.*

In the proof below we use the following two facts:

(a) If $P = [x_1, x_2, x_3]$ and $Q = [y_1, y_2, y_3]$ are two different Points in $\mathbb{R}P^2$, then the Line $PQ$ is given by

$$PQ = \{[z_1, z_2, z_3] \in \mathbb{R}P^2 \mid az_1 + bz_2 + cz_3 = 0\},$$

where

$$\begin{aligned}(a, b, c) &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (\det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}, -\det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix}, \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.\end{aligned}$$

(b) If $l_1 = \{a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ and $l_2 = \{b_1x_1 + b_2x_2 + b_3x_3\}$ are two different Lines in $\mathbb{R}P^2$, then the intersection Point $P = l_1 \cap l_2$ has the homogeneous coordinates $P = [z_1, z_2, z_3]$ given by

$$(z_1, z_2, z_3) = (a_1, a_2, a_3) \times (b_1, b_2, b_3).$$

*Proof.* By Lemma 3.24 we can assume that

$$P_1 = [1, 0, 0], \quad Q_1 = [0, 0, 1], \quad Q_2 = [1, 1, 1]$$

and $S = [0, 1, 0]$, where $S$ is the intersection Point of the tangents at $P_1$ and $Q_1$ and that $C$ is given by

$$x_2^2 - x_1x_3 = 0.$$

Since the tangent $Q_1S$ has the form $\{[x_1, x_2, x_3] \mid x_1 = 0\}$, by Lemma 3.23(b) none of the Points $P_2, P_3, Q_3$ has vanishing first homogeneous coordinate. Therefore, there exist $r, s, t \in \mathbb{R}\backslash\{0\}$ such that

$$P_2 = [1, r, r^2], \quad P_3 = [1, s, s^2], \quad Q_3 = [1, t, t^2],$$

and $r, s, t$ are pairwise different and none of them equals 1. We have $P_1Q_2 = \{x_2 = x_3\}$ and $Q_1P_2 = \{-rx_1 + x_2 = 0\}$ since

$$(0, 0, 1) \times (1, r, r^2) = (-r, 1, 0).$$

This implies that

$$P_1Q_2 \cap Q_1P_2 = (1, r, r) = R_3.$$

Similarly, we obtain

$$\begin{aligned}
P_1Q_3 &= \{-tx_2 + x_3 = 0\}, \quad \text{since } (1,0,0) \times (-1,t,t^2) = (0,-t^2,t), \\
Q_1P_3 &= \{-sx_1 + x_2 = 0\}, \quad \text{since } (0,0,1) \times (1,s,s^2) = (-s,1,0), \\
P_2Q_3 &= \{-rtx_1 + (r+t)x_2 - x_3 = 0\}, \\
&\quad \text{since } (1,r,r^2) \times (1,t,t^2) = (r-t)(-rt, r+t, -1), \\
Q_2P_3 &= \{-sx_1 + (s+1)x_2 - x_3 = 0\}, \\
&\quad \text{since } (1,1,1) \times (1,s,s^2) = (1-s)(-s, s+1, -1).
\end{aligned}$$

This implies that $R_2 = P_1Q_3 \cap Q_1P_3 = (1, s, st)$ and $R_3 = P_1Q_2 \cap Q_1P_2 = [z_1, z_2, z_3]$ with

$$(z_1, z_2, z_3) = (-rt, r+t, -1) \times (-s, s+1, -1) = (1+s-r-t, s-rt, sr+st-rt-rst).$$

Now,

$$\begin{pmatrix} 1+s-r-t \\ s-rt \\ sr+st-rt-rst \end{pmatrix} = \begin{pmatrix} (1-r)+(s-t) \\ s-rt \\ (1-r)st + r(s-t) \end{pmatrix} = (1-r)\begin{pmatrix} 1 \\ s \\ st \end{pmatrix} + (s-t)\begin{pmatrix} 1 \\ r \\ r \end{pmatrix},$$

i.e., the three Points $R_1, R_2$ and $R_3$ are collinear. □

**Corollary 3.26.** *Given five distinct Points $P_1 = [z_1], \dots P_5 = [z_5] \in \mathbb{R}P^2$, no four of them collinear, then there exists a unique conic $C$ passing through them.*

*Proof. Existence:* The five conditions

$$z_j^\top \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} z_j = 0$$

yield five homogeneous linear equations for $a, b, c, d, e, f$. This implies that there exists a non-trivial solution.

*Uniqueness:* If three Points, e.g., $P_1, P_2, P_3$ are collinear, then $C$ is singular by Lemma 3.23(a) and consists of two Lines be the classification result for conics (see bottom page 36). The second Line is uniquely determined by the remaining two Points $P_4, P_5$.

Assume no three of the Points $P_1, \dots, P_5$ are collinear. Then any conic $C$ through $P_1, \dots, P_5$ is non-singular. For everty Line $l$ through $P_1$ not tangent to $C$, the second intersection Point $P_6$ of $l \cap C$ can be uniquely determined by Pascal's Theorem: Let $Q_1 = P_4, Q_2 = P_5$.

(a) Let $R_3 = P_1Q_2 \cap Q_1P_2$.

(b) Let $R_2 = l \cap P_3Q_1$.

(c) Let $l_R = R_2R_3$.

(d) Let $R_1 = l_R \cap P_3Q_2$.

Finally, we obtain $P_6$ as the intersection Point $l \cap P_2R_1$. In this way, we construct the unique conic $C$ through the Points $P_1, \dots, P_5$. □