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Geometry III/IV

Solutions, set 1.

Exercise 1. (a) We have

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle + \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle \\ &= 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

Since $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$, we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

(b) Choose $f(x) = 1-x$ and $g(x) = x$. Then $f(x)+g(x) = 1$ and $f(x)-g(x) = 1-2x$ and, therefore, $\|f+g\|_\infty = 1$ and $\|f-g\|_\infty = 1$. But $\|f\|_\infty = \|g\|_\infty = 1$, and we get

$$2 = \|f+g\|^2 + \|f-g\|^2 < 2\left(\|f\|^2 + \|g\|^2\right) = 4.$$

Exercise 2. (a) We have to show that

$$|\sum x_i y_i| \le \left(\sum x_i^2\right)^{1/2}\left(\sum y_i^2\right)^{1/2}.$$

Squaring here is an equivalence relation, so we have to show that

$$\left(\sum x_i y_i\right)^2 \le \left(\sum x_i^2\right)\left(\sum y_i^2\right).$$

This is equivalent to

$$\sum_i (x_i y_i)^2 + \sum_{i\neq j} x_i y_i x_j y_i \le \sum_i x_i^2 y_i^2 + \sum_{i\neq j} x_i^2 y_j^2,$$

and to
$$0 \le \sum_{i\le j}(x_i^2 y_j^2 + x_j^2 y_i^2 - 2x_i y_i x_j y_j) = \sum_{i\le j}(x_i y_j - x_j y_i)^2,$$
finishing the proof of Cauchy-Schwarz.

(b) The triangle inequality $d(x,z) + d(z,y) \ge d(x,y)$ translates into
$$\|x-z\| + \|z-y\| \ge \|x-y\|.$$

Setting $u = x - z$ and $v = z - y$, this is equivalent to
$$\|u\| + \|v\| \ge \|u+v\| \quad \forall\, u,v \in \mathbb{R}^n.$$
Squaring is here, again, an equivalence relation, so we obtain
$$\|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2 \ge \|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u,v\rangle.$$
This reduces to Cauchy-Schwarz, namely,
$$\|u\|\cdot\|v\| \ge \langle u,v\rangle.$$

Exercise 3. (a) Let $V \subset \mathbb{R}^2$ the linear subspace parallel to l_1 and l_2, so $l_2 = \lambda v + V$ and $l_1 = (\lambda+1)v + V$ with an appropriate $\lambda \in \mathbb{R}$. Note that s_V is linear, i.e., $s_V(x) = S_\alpha x$ with $\alpha \in [0,2\pi)$ and, since $v \perp V$ and $s_V^{-1} = s_V$, we conclude from Theorem 1.13 that $s_V \circ t_{-v} \circ s_V = t_v$. Note that
$$\begin{aligned} s_{l_2} &= t_{\lambda v} \circ s_V \circ t_{-\lambda v}, \\ s_{l_1} &= t_{(\lambda+1)v} \circ s_V \circ t_{-(\lambda+1)v}. \end{aligned}$$
Consequently,
$$s_{l_1} \circ s_{l_2} = t_{(\lambda+1)v} \circ (s_V \circ t_{-v} \circ s_V) \circ t_{-\lambda v} = t_{(\lambda+1)v} \circ t_v \circ t_{-\lambda v} = t_{2v}.$$

(b) Note that $l_1 = r_{\alpha,z}(l_2)$. We conclude from Theorem 1.13 that
$$r_{\alpha,z} \circ s_{l_2} \circ r_{-\alpha,z} = s_{l_1}.$$
Since $s_{l_2}^{-1} = s_{l_2}$, we conclude again from Theorem 1.13 that
$$s_{l_1} \circ s_{l_2} = r_{\alpha,z} \circ (s_{l_2} \circ r_{-\alpha,z} \circ s_{l_2}) = r_{\alpha,z} \circ r_{\alpha, s_{l_2}(z)} = r_{\alpha,z} \circ r_{\alpha,z} = r_{2\alpha,z}.$$

(c) From (a) and (b) we see that translations and general rotations can be written as the composition of two reflections. Since a glide-reflection is the composition of a reflection and a translations, it can be written as the composition of three reflections. Glide reflections are the only isometries which cannot be written by less than three reflections, but three reflections are needed since glide reflections don't agree with reflections (reflections have non-empty fix point sets and the fixed point set of a glide reflection is empty), so glide reflections cannot be written as a single reflection, and the composition of two reflections is orientation preserving, whereas glide reflections are orientation reversing.