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Geometry III/IV

Solutions, set 2.

Exercise 4. (a) Centres of rotational symmetry must be either (i) a vertex of a tile (equals intersection point of a vertical and a horizontal line), (ii) a midpoint of a side of a tile, or (iii) the centre of a tile. In case (i), the pattern is mapped onto itself iff $\alpha = k \cdot \frac{\pi}{2}$, $k = 0, 1, 2, 3$. In case (ii), only the full turn maps the pattern onto itself, and in case (iii), the pattern is mapped onto itself iff $\alpha = k \cdot \pi$, $k = 0, 1$.

Reflectional symmetry: Every line l of reflection intersects with all horizontal lines or with all vertical lines of the pattern. Assume that l intersects with all horizontal lines. If l intersects a horizontal line at q, and q is not a vertex of the pattern, then the reflection at l must obviously map the horizontal line onto itself. This is only possible if l is vertical and passes through a midpoint of a horizontal side of a tile. In this case, the pattern is mapped onto itself. If l intersects a horizontal line at a vertex q, then it might be, at first sight, that l maps the horizontal line through q to the vertical line through q of the pattern, but one easily sees that this is not possible. Neither can a reflection along a line through a vertex q map the horizontal line through q onto itself. The same arguments hold for reflection lines intersecting all vertical lines of the pattern. So the reflectional symmetries is precisely the reflections along horizontal or vertical lines through midpoints of horizontal or vertical sides of tiles.

(b) For a given tile $\mathcal{T}$, let p be its lower left vertex. A translation t_v in Γ must map p to a lower left vertex of a tile, and any lower left vertex of a tile can be reached by a translation in Γ. Let t_{v_1} map the lower left vertex to the lower right vertex and t_{v_2} map the lower left vertex to the upper left vertex. Then all lower left vertices are precisely at the positions $p + nv_1 + mv_2$ with $n + m \equiv 0 \mod 2$. Thus we have

$$T(\Gamma) = \{t_{nv_1+mv_2} \mid n + m \equiv 0 \mod 2\}.$$

(c) We first try to understand Γ: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry in Γ. Then $f(p)$ is a vertex of the pattern. By a translation $t_{2nv_1+2mv_2}$, $f(p)$ can be translated to a vertex of the tile $\mathcal{T}$. If this is not the lower left vertex, it can be mapped to the lower left vertex by appropriate applications of the horizontal or vertical reflections s_h or s_v along the middle of the vertical or horizontal sides of $\mathcal{T}$. Let g be this composition of $t_{2nv_1+2mv_2}$, s_h and s_v with $g \circ f(p) = p$. By (a), this isometry must be $r_{\frac{k\pi}{2},p}$ for some $k = 0, 1, 2, 3$. Thus generators of Γ are the isometries $t_{2v_1}, t_{2v_2}, s_h, s_v$ and $r_{\frac{\pi}{2},p}$. You can drop t_{2v_2} from this list, since you obtain it via the conjugation $r_{\frac{\pi}{2},p} \circ t_{2v_1} \circ r_{-\frac{\pi}{2},p}$.

(d) The linear parts of the generators are $I, R_{\frac{k\pi}{2}}, S_0, S_\pi$. They generate the derived group. By the remark after Lemma 1.12, we conclude that

$$\Gamma' = \{R_{\frac{k\pi}{2}} \mid k = 0, 1, 2, 3\} \cup \{S_{\frac{k\pi}{2}} \mid k = 0, 1, 2, 3\}.$$

Exercise 5. Obviously, discreteness of Γ implies the "ball property". Now, assume that the "ball property" is satisfied. For arbitrarily chosen $x_0 \in \mathbb{R}^n$ and bounded $B \subset \mathbb{R}^n$, choose

$$r := d(x_0, 0) + \sup_{x \in B}(x, 0) < \infty.$$

Then $f(x_0) \in B$ implies that

$$d(f(0), 0) \leq d(f(0), f(x_0)) + d(f(x_0), 0) = d(0, x_0) + d(f(x_0), 0) \leq r,$$

and thus

$$|\{f \in \Gamma \mid f(x_0) \in B\}| \leq |\{f \in \Gamma \mid f(0) \in B_r\}| < \infty.$$

Exercise 6. Let $t_a \in T(\Gamma)$ and $f \in \Gamma$. We have to show that $g = f \circ t_a \circ f^{-1} \in T(\Gamma)$. We know from Theorem 1.13 that $g = t_{f'(a)}$ and from Lemma 1.18 that $t_{f'(a)} \in T(\Gamma)$. This shows that $T(\Gamma)$ is normal in G. Let us look at the map

$$\Phi : \Gamma \to \Gamma', \qquad \Phi(f) = f'.$$

By Lemma 1.12, this is a surjective group homomorphism. Obviously, the kernel of Φ coincides with the translation group Γ, since $\Phi(f) = \mathrm{Id}_{\mathbb{R}^n}$ is equivalent to $f(x) = x + b$ for suitable $b \in \mathbb{R}^n$, i.e., $f \in T(\Gamma)$. Then the image of Φ, namely Γ', is isomorphic to $\Gamma/\mathrm{ker}\Phi = \Gamma/T(\Gamma)$.