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Geometry III/IV

Solutions, set 3.

Exercise 7. Let $\mathcal{T}$ be a tile of the pattern with lower left corner $p \in \mathbb{R}^2$ and l_h, l_v denote the horizontal and vertical lines passing through the middles of the vertical and horizontal sides of $\mathcal{T}$. Let s_h, s_v denote the corresponding reflections along l_h and l_v. Then l_h and l_v divide $\mathcal{T}$ into four (open) parts and we denote the lower left part by $\mathcal{F}$. We show that $\mathcal{F}$ is a fundamental domain.

(a) Proof that $\bigcup_{g \in \Gamma} \overline{g(\mathcal{F})} = \mathbb{R}^2$: We first see that

$$\overline{\mathcal{T}} = \overline{\mathcal{F}} \cup \overline{s_h(\mathcal{F})} \cup \overline{s_v(\mathcal{F})} \cup \overline{(s_v s_h)(\mathcal{F})}.$$

Taking the union of the rotations $r_{\frac{k\pi}{2}}(\overline{\mathcal{T}})$, we cover the closure of four tiles, arranged in a square. Finally, by applying the translations t_{2v_1}, t_{2v_2}, we end up covering all of $\mathbb{R}^2$.

(b) Proof that for $g \neq e$, $g(\mathcal{F})$ and $\mathcal{F}$ are disjoint: Since Γ maps tiles to tiles, $g(\mathcal{F})$ must lie completely in a tile of our pattern. If this is another tile, then $\mathcal{F}$ and $g(\mathcal{F})$ are disjoint. So let us assume that $g(\mathcal{F}) \subset \mathcal{T}$. The closure $\overline{\mathcal{F}}$ contains exactly one vertex of the pattern, namely p. So $\overline{g(\mathcal{F})}$ must also contain precisely one vertex of the pattern, and if this is vertex different to p of the tile $\mathcal{T}$, then $g(\mathcal{F})$ is another one of the four parts of $\mathcal{F}$ and cannot intersect with $\mathcal{F}$. Therefore,

$$p \in \overline{\mathcal{F}} \cap \overline{g(\mathcal{F})}.$$

Thus, we must have $p = g(p)$. The only isometries $f \in \Gamma$ fixing p are the rotations $r_{\frac{k\pi}{2}}$ and reflections along lines through p, but there are no such reflections in Γ. The non-trivial rotations obviously map $\mathcal{F}$ onto disjoint new sets.

(c) Proof that that there are only finitely many $g \in \Gamma$ such that $\overline{g(\mathcal{F})}$ intersects with $\overline{\mathcal{F}}$: One easily sees that intersections can only occur with the other three parts of the tile $\mathcal{T}$ and with five images in the three adjacent tiles. So there are in total only 8 "neighbours" of $\mathcal{F}$.

Exercise 8.

(a) Since G is finite, there is a minimal $k \geq 2$ such that $r^k = e$. This implies that $k\alpha \in 2\pi\mathbb{Z}$, i.e., $\alpha = \frac{2\pi l}{k}$. Minimality of k implies that $gcd(k,l) = 1$. Consequently, there are integers $p, q \in \mathbb{Z}$ with $pk + ql = 1$ and we conclude that $r^q = R_{\frac{2\pi}{k}}$. Minimality of α implies then that $\alpha = \frac{2\pi}{k}$. The group G contains then all rotations of the form $R_{\frac{2\pi l}{k}}$ with $0 \leq l \leq k-1$. If G would contain a rotation R_β with $\beta \in (0, 2\pi)$ and $\beta \neq \frac{2\pi l}{k}$ for $l = 0, 1, \ldots, k-1$, then there would be a $l \in \{0, \ldots, k-1\}$ with

$$\frac{2\pi l}{k} < \beta < \frac{2\pi(l+1)}{k}.$$

But then $R_\beta R_{-\frac{2\pi l}{k}} = R_\gamma \in G$ with $0 < \gamma < \frac{1\pi}{k}$ in contradiction to the minimality of α. Then G coincides with the group $\{R_{\frac{2\pi l}{k}} \mid l = 0, 1, \ldots, k-1\}$ and is thus isomorphic to C_k.

(b) Obviously, G_0 are the orientation preserving elements of G and G_1 are the orientation reversing elements of G. If $g \in G_0$ then sg is orientation reversing and hence $f(g) = sg \in G_1$. If $g \in G_1$ then sg is orientation preserving and hence $f(g) = sg \in G_0$. $f : G_0 \to G_1$ is injective, since f^2 is the identity. $f : G_0 \to G_1$ is surjective since for $g \in G_1$, $h = f(g) \in G_0$ and $f(h) = f^2(g) = g$. Thus $f : G_0 \to G_1$ is bijective and $|G_0| = |G_1|$. Since G is the disjoint union of G_0 and G_1, we have $|G| = 2|G_0|$. Let $r = R_\alpha$ be the generator of $G_0 \subset SO(2)$ with minimal angle $\alpha = 2\pi/k > 0$, i.e., $G_0 = \{r^j \mid j = 0, 1, \ldots, k-1\}$. Then we have

$$G = G_0 \cup G_1 = \{e, r, \ldots, r^{k-1}, f(e), f(r), \ldots, f(r^{k-1})\},$$

finishing the proof of (b). Since $s \in O(2)$ is orientation reversing, it must be a reflection along a line l, and since all elements in $O(2)$ fix the origin, we must have $0 \in l$.

(c) Let We conlude from Theorem 1.13 that

$$srs = sR_\alpha s^{-1} = R_{-\alpha} = r^{-1}.$$

Since G has $2k$ elements and satisfies the same relations like the dihedral group D_k with $2k$ elements, we conclude that G is isomorphic to D_k.