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Geometry III/IV

Solutions, set 4.

Exercise 9. (a) Let us denote the coordinates of π_2 by x', y', to avoid confusion. One easily checks that

$$(x', y') = f(x, y) = (x, y/\sin\alpha).$$

Assume a circle in $\subset \pi_1$ is given by the equation $(x - x_0)^2 + (y - y_0)^2 = r^2$, $r > 0$. All points of this circle are of the form $(x_0 + r\cos\gamma, y_0 + r\sin\gamma)$, which are mapped to

$$(x', y') = f(x_0 + r\cos\gamma, y_0 + r\sin\gamma) = (x_0 + r\cos\gamma, \frac{y_0}{\sin\alpha} + \frac{r}{\sin\alpha}\sin\gamma).$$

The image points, therefore, are solutions of the equation

$$(x' - x_0)^2 + (y' - \frac{y_0}{\sin\alpha})^2(\sin\alpha)^2 = r^2\cos^2\gamma + r^2\sin^2\gamma = r^2,$$

which translates into

$$\left(\frac{x' - x_0}{r}\right)^2 + \left(\frac{y' - \frac{y_0}{\sin\alpha}}{r/\sin\alpha}\right)^2 = 1,$$

with $r, r/\sin\alpha > 0$.

(b) Using (a), we can place the ellipse in plane π_2 and choose the angle $\alpha \in (0, \pi/2]$ appropriately to have a circle C as the preimage in π_1 under $f : \pi_1 \longrightarrow \pi_2$. Parallel chords of the ellipse are mapped to parallel chords of the circle and midpoints of these chords of the ellipse are mapped to midpoints of the corresponding chords in the circle. Think for the moment of our circle to have its centre in the origin and the parallel chords of the circle to be vertical line segments. By symmetry, their midpoints then lie on the x-axis. This immediately implies that the midpoints of parallel chords in a general circle lie always on a line. This straight line is mapped under f to a straight line in π_2, which contains all the midpoints of the parallel chords of the ellipse. Note that we used the following properties of parallel projections:

- straight lines are mapped to straight lines
- parallel lines are mapped to parallel lines
- ratios along lines are preserved

Exercise 10.

(a) A straightforward calculation shows that

$$f(x) = \begin{pmatrix} -3 & -1 \\ -6 & -4 \end{pmatrix} x + \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

and

$$f^{-1}(x) = \frac{1}{6} \begin{pmatrix} -4 & 1 \\ 6 & -3 \end{pmatrix} x + \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}.$$

(b) A straightforward calculation shows that

$$g(x) = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) The affine transformation can be constructed via $g \circ f^{-1}$. A straightforward calculation yields

$$(g \circ f^{-1})(x) = \frac{1}{3} \begin{pmatrix} 8 & -5 \\ 3 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$