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Geometry III/IV

Solutions, set 5.

Exercise 11.

For the fixed points of f we have to solve the equation $f(x) = x$, i.e.,

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} x = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

One easily sees that the set of fixed points is the line

$$\{\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \end{pmatrix} \mid \lambda \in \mathbb{R}\}.$$

The fixed point equation $g(x) = x$ leads to

$$\begin{pmatrix} 4 & 2 \\ -6 & -3 \end{pmatrix} x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since the second row of this matrix is $-3/2$ times the first row, any vector of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -6 & -3 \end{pmatrix} x$$

has the property that $b = -3/2a$. This shows that g doesn't have any fixed points.

Now we calculate the eigenvalues of f:

$$p(t) = \det \begin{pmatrix} t-4 & -9 \\ 1 & t+2 \end{pmatrix} = (t-4)(t+2) + 9 = t^2 - 2t + 1 = (t-1)^2,$$

i.e., we have a double eigenvalue $\lambda = 1$. Since $\ker f$ is one-dimensional and f has fixed points, we conclude that f is conjugate to the normal form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x.$$

The eigenvalues of g are given by

$$p(t) = \det\det \begin{pmatrix} t-5 & -2 \\ 6 & t+2 \end{pmatrix} = (t+5)(t+2)+12 = t^2-3t+2 = (t-1)(t-2),$$

i.e., we have the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$. Since g has no fixed points, it is conjugate to the normal form

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Exercise 12. By the Fundamental Theorem of Affine Geometry, there is an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(A) = 0$, $f(B) = e_1$, $f(C) = e_2$. The image $f(l)$ cannot be vertical, since it wouldn't cross the three extended sides of the standard triangle in this case. Therefore, $f(l)$ is a line given by an equation $y = mx + b$ where $m \in \mathbb{R}$ is the slope of this line. Since affine transformations prevere ratios, it suffices to prove Menelaus' theorem in the case of this line $y = mx + b$ crossing the extended sides of the standard triangle ΔABC with $A = 0$, $B = e_1$ and $C = e_2$. We have

$$\begin{aligned} R = l \cap l_{AB} &= \left(-\frac{b}{m}, 0\right), \\ P = l \cap l_{BC} &= \left(\frac{1-b}{1+m}, \frac{b+m}{1+m}\right), \\ Q = l \cap l_{CA} &= (0, b). \end{aligned}$$

Using the x- and y-coordinate formulas, we obtain

$$\begin{aligned} &\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} \\ &\qquad = \frac{0-(-b/m)}{-(b/m)-1} \cdot \frac{(1-b)/(1+m)-1}{0-(1-b)/(1+m)} \cdot \frac{b-1}{0-b} \\ &\qquad\qquad = \frac{b}{-b-m} \cdot \frac{-b-m}{b-1} \cdot \frac{b-1}{-b} = -1 \end{aligned}$$