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Geometry III/IV

Solutions, set 6.

Exercise 13. In picture (a) we apply Menelaus' Theorem and obtain:

$$-1 = \frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{-(2+x)}{x} \cdot \frac{1}{1} \cdot \frac{1}{2} = -\frac{2+x}{2x},$$

which implies that $2 + x = 2x$ and $x = 2$.

In picture (b) we apply Ceva's Theorem and obtain

$$1 = \frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{-(y+1)}{1} \cdot \frac{2}{2} \cdot \frac{-1}{4} = \frac{y+1}{4},$$

which implies that $y + 1 = 4$ and $y = 3$.

Exercise 14. One easily checks that any three of the four vectors

$$v_a = (1,0,2),\ v_b = (0,1,0),\ v_c = (1,-1,0),\ v_d = (2,1,1)$$

are linearly independent. Therefore, no three of the four points A, B, C, D are collinear. $X = [x_1, x_2, x_3] \in AB$ iff $(x_1, x_2, x_3) \neq 0$ and

$$0 = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = -2x_1 + x_3.$$

Similarly, $X \in CD$ iff $(x_1, x_2, x_3) \neq 0$ and

$$0 = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = -x_1 - x_2 + 3x_3.$$

A non-trivial simultaneous solution of both equations is $(x_1, x_2, x_3) = (1, 5, 2)$ and we conclude that $AB \cap CD = [1, 5, 2]$.

Exercise 15. (a) We have

$$\begin{aligned}\|\Phi(z,w)\|^2 &= \frac{4(\mathrm{Re}(z\bar{w})^2+\mathrm{Im}(z\bar{w})^2)+(|z|^2-|w|^2)^2}{(|z|^2+|w|^2)^2}\\ &= \frac{4|zw|^2+(|z|^4-2|zw|^2+|w|^4)}{(|z|^2+|w|^2)^2} = \frac{(|z|^2+|w|^2)^2}{(|z|^2+|w|^2)^2} = 1.\end{aligned}$$

(b) Note that $(\lambda z)\overline{\lambda w} = |\lambda|^2 z\bar{w}$. Using this fact, we see, comparing each component of $\Phi(\lambda z, \lambda w)$ with the corresponding component of $\Phi(z,w)$, that both numerator and denominator are differing by the factor $|\lambda|^2$, which cancels out.

(c) For injectivity, we assume that $\Phi(z,w) = \Phi(u,v)$. This implies that we have the identities

$$\frac{\mathrm{Re}(z\bar{w})}{|z|^2+|w|^2} = \frac{\mathrm{Re}(u\bar{v})}{|u|^2+|v|^2}, \tag{1}$$

$$\frac{\mathrm{Im}(z\bar{w})}{|z|^2+|w|^2} = \frac{\mathrm{Im}(u\bar{v})}{|u|^2+|v|^2}, \tag{2}$$

$$\frac{|z|^2-|w|^2}{|z|^2+|w|^2} = \frac{|u|^2-|v|^2}{|u|^2+|v|^2}. \tag{3}$$

Note that $[z,w],[u,v] \in \mathbb{C}P^1$ implies that $(z,w) \neq 0 \neq (u,v)$ and thus $|z|^2+|w|^2 > 0$ and $|u|^2+|v|^2 > 0$. If $w=0$, we conclude from (1) and (2) that either u or v must be zero. But (3) forces v to be zero, which means that we have $[z,w] = [z,0] = [u,0] = [u,v]$. Similarly, one concludes that if $z=0$ we must also have $u=0$ and thus $[z,w] = [u,v]$. So we can now assume that $z,w,u,v \neq 0$ and therefore, that there are non-zero complex numbers $a,b \in \mathbb{C}$ such that $u = az$ and $v = bw$. Multiplying up (3), we conclude that

$$|zv|^2 - |wu|^2 = -|zv|^2 + |wu|^2 = 0.$$

This yields $|zv| = |wu|$ or

$$|azw| = |bzw| \Longrightarrow |a| = |b|.$$

Hence, a,b have the same modulus and we can write $a = re^{i\alpha}$ and $b = re^{i\beta}$ with $r > 0$. Using (1), we conclude

$$\begin{aligned}\frac{\mathrm{Re}(z\bar{w})}{|z|^2+|w|^2} &= \frac{r^2\mathrm{Re}(e^{i(\alpha-\beta)}z\bar{w})}{r^2(|z|^2+|w|^2)},\\ \mathrm{Re}(z\bar{w}) &= \mathrm{Re}(e^{i(\alpha-\beta)}z\bar{w}).\end{aligned}$$

Similarly, we obtain $\mathrm{Im}(z\bar{w}) = \mathrm{Im}(e^{i(\alpha-\beta)}z\bar{w})$, and thus

$$z\bar{w} = e^{i(\alpha-\beta)}z\bar{w},$$

which implies $e^{i(\alpha-\beta)} = 1$ and thus $a = b$. This yields

$$[u, v] = [az, aw] = [z, w],$$

finishing the proof of injectivity.

(d) We see that $(0, 0, 1) \in S^2$ lies in the image of Φ_0. Now note that the image of the map $r \mapsto \frac{r^2-1}{r^2+1}$ for $r \geq 0$ covers all of $[-1, 1)$. Let $(X, Y, Z) \in S^2 \backslash \{(0, 0, 1)\}$ be given. Then we can find $r_0 \geq 0$ such that

$$\frac{r_0^2 - 1}{r^2 + 1} = Z.$$

Let $z = r_0 e^{i\alpha}$. Since $\Phi_0([z, 1]) \in S^2$ and the third component of $\Phi_0([z, 1])$ coincides with Z, we know that

$$\left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}\right) \quad \text{and } (X, Y)$$

lie on the same circle C about the origin in $\mathbb{R}^2$. Since

$$\left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}\right) = \left(\frac{2r_0 \cos\alpha}{r_0^2 + 1}, \frac{2r_0 \sin\alpha}{r_0^2 + 1}\right),$$

we can reach every point on this circle C by choosing $\alpha \in [0, 2\pi)$ appropriately. So the right choice of α yields $(X, Y, Z) = \Phi_0([r_0 e^{i\alpha}, 1])$, finishing the proof of surjectivity.