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Geometry III/IV

Solutions, set 7.

Exercise 16. The line P_1P_2 corresponds to the plane $E_P := \{\lambda u_1 + \mu u_2\}$ with normal vector $u_1 \times u_2$. Let E_Q and E_R be similarly defined. Concurrency of the three lines P_1P_2, Q_1Q_2 and R_1R_2 coincides with the fact that the intersection $E_P \cap E_Q \cap E_R$ is a straight line l through the origin. This is precisely the case if the three normal vectors $u_1 \times u_2$, $v_1 \times v_2$ and $w_1 \times w_2$ lie in a common plane $F \subset \mathbb{R}^3$ through the origin and the line normal to F through the origin agrees with l. $u_1 \times u_2$, $v_1 \times v_2$ and $w_1 \times w_2$ lie in a common plane if the matrix built by these vectors as columns is singular, i.e., if its determinant equals zero.

Exercise 17.

(a) Let $v_0 = (1,2,0)$, $v_1 = (0,1,0)$, $v_2 = (-1,0,2)$, $v_3 = (-1,3,4)$ and $w_0 = (1,0,0)$, $w_1 = (1,1,0)$, $w_2 = (0,-1,1)$, $w_3 = (1,1,1)$. Then we have

$$\begin{aligned} v_3 &= v_0 + v_1 + 2v_2, \\ w_3 &= -w_0 + 2w_1 + w_2. \end{aligned}$$

Set $v_0' = v_0 = (1,2,0)$, $v_1' = v_1 = (0,1,0)$, $v_2' = 2v_2 = (-2,0,4)$, $v_3' = v_3 = (-1,3,4)$ and $w_0' = -w_0 = (-1,0,0)$, $w_1' = 2w_1 = (2,2,0)$, $w_2' = w_2 = (0,-1,1)$, $w_3' = w_3 = (1,1,1)$. We conclude that

$$\begin{aligned} v_3' &= v_0' + v_1' + v_2', \\ w_3' &= w_0' + w_1' + w_2'. \end{aligned}$$

Then the linear map A satisfying $Av_j' = w_j'$ for $j = 0,1,2$ also satisfies $Av_3' = w_3'$. A is given by

$$\begin{aligned} A = &\begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 4 & 0 & 2 \\ -8 & 4 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 & -5/2 \\ -4 & 2 & 9/4 \\ 0 & 0 & 1/4 \end{pmatrix}. \end{aligned}$$

Multiplication of A by an arbitrary factor doesn't change the induced projective transformation, so we define

$$f([x]) := \left[\begin{pmatrix} -20 & 8 & -10 \\ -16 & 8 & -9 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right].$$

(b) The matrix

$$B := \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

maps $e_1, e_2, e_3, e_1 + e_2 + e_3$ to $-e_2, -e_3, e_1 + e_2 + e_3, e_1$, respectively. Using (a), we conclude that the required matrix is given by

$$\begin{aligned} C &= \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} B \underbrace{\begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}^{-1}}_{=D} \\ &= \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} B \begin{pmatrix} 1 & 0 & 1/2 \\ -2 & 1 & -1 \\ 0 & 0 & 1/4 \end{pmatrix} = \begin{pmatrix} -4 & 2 & -9/4 \\ -1 & 0 & 1/4 \\ 8 & -4 & 5 \end{pmatrix} \end{aligned}$$

since the projective map corresponding to the matrix D maps P_0, P_1, P_2, P_3 to $[e_1], [e_2], [e_3], [e_1 + e_2 + e_3]$, respectively. Consequently, we define

$$g([x]) := \left[\begin{pmatrix} -16 & 8 & -9 \\ -4 & 0 & 1 \\ 32 & -16 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right].$$

(c) From

$$(4C) \cdot (4A) = \begin{pmatrix} 192 & -64 & 79 \\ 80 & -32 & 41 \\ -384 & 128 & -156 \end{pmatrix} = E$$

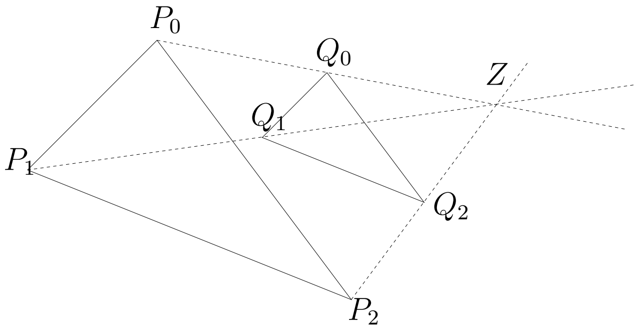
we conclude that

$$g \circ f([x]) = [Ex].$$

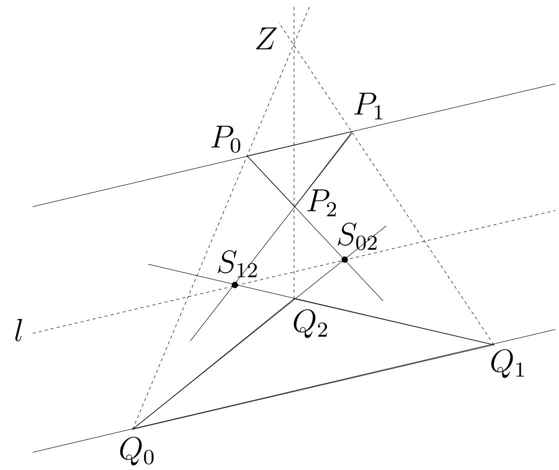
Exercise 18. By Desargues Theorem of Projective Geometry we know that the three intersection points $S_{ij} = P_iP_j \cap Q_iQ_j$ for $i, j \in \{0, 1, 2\}$ are collinear. We distinguish the following cases:

(a) None of the S_{ij} is infinitely far away: This means that all pairs P_iP_j, Q_iQ_j of lines intersect in $\mathbb{R}^2$ and lie on a common affine line.

(b) Assume that two of the three projective intersection points, say S_{01} and S_{02} are infinitively far away. Then the projective line through S_{01} and S_{02} is the infinitely far away line l_∞ and $S_{12} \in l_\infty$, by Desargues's Theorem. This translates (in the affine world) to the case that all three pairs of lines P_iP_j, Q_iQ_j are parallel.



(c) Assume that only one the three projective intersection points is infinitely far away, say S_{01}. Then the other two finite intersection points S_{02} and S_{12} determine an affine line l and S_{01} has to be the uniquely defined infinitely far away point of this line (by Desargues), i.e., P_0P_1 and Q_0Q_1 must be parallel to l and to each other.



Exercise 19. The projective line P_1P_2 determines a Euclidean plane $E_P \subset \mathbb{R}^3$ through the origin with normal vector $n_1 := v_1 \times v_2$. Similarly, Q_1Q_2 determines a Euclidean plane $E_Q \subset \mathbb{R}^3$ through the origin with normal vector $n_2 := w_1 \times w_2$. Note that both vectors n_1, n_2 cannot be collinear, for otherwise E_P and E_Q would coincide and Q_1 would lie in the projective line P_1P_2. The intersection $E_P \cap E_Q$ is a line $l \subset \mathbb{R}^3$ through the origin. We show that $n_1 \times n_2$ spans this line. Since $n_1 \perp E_P$ and $(n_1 \times n_2) \perp n_1$ (since n_1, n_2 are linearly independent), we conclude that $(n_1 \times n_2) \in E_P$ for dimension reasons. Similarly, we conclude that $(n_1 \times n_2) \in E_Q$, and therefore we have

$$0 \neq n_1 \times n_2 \in E_P \cap E_Q = l.$$

This shows that

$$[n_1 \times n_2] = [(v_1 \times v_2) \times (w_1 \times w_2)] \in P^2(\mathbb{R})$$

is the intersection point of $P_1P_2 \cap Q_1Q_2$.