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Geometry III/IV

Solutions, set 8.

Exercise 20. *Brianchon's Theorem:* Let l_0, l_1, l_2 be three concurrent Lines in $\mathbb{R}P^2$ and l'_0, l'_1, l'_2 be three concurrent Lines with a different intersection Point. For $0 \le i < j \le 2$, let l_{ij} be the Line joining $l_i \cap l'_j$ and $l'_i \cap l'_j$. Then the three Lines l_{01}, l_{02} and l_{12} are concurrent.

Exercise 21.

(a) Let $p_i = [v_i]$ and assume that

$$v_2 = \alpha v_0 + \beta v_1, \qquad v_3 = \gamma v_0 + \delta v_1.$$

Then we have

$$[p_0, p_1; p_2, p_3] = \frac{\beta}{\alpha} \Big/ \frac{\delta}{\gamma}.$$

Moreover, we have

$$v_0 = -\frac{\beta}{\alpha} v_1 + \frac{1}{\alpha} v_2$$

and

$$v_3 = \frac{\gamma}{\alpha}(\alpha v_0 + \beta v_1) + (\delta - \frac{\gamma}{\alpha}\beta) v_1 = \frac{\alpha\delta - \beta\gamma}{\alpha} v_1 = \frac{\gamma}{\alpha} v_2.$$

This implies that

$$\begin{aligned}
[p_1, p_2; p_0, p_3] &= \frac{1/\alpha}{-\beta/\alpha} \Big/ \frac{\gamma/\alpha}{(\alpha\delta - \beta\gamma)/\alpha} \\
&= -\frac{\alpha\delta - \beta\gamma}{\beta\gamma} \\
&= 1 - \frac{1}{\frac{\beta}{\alpha}/\frac{\delta}{\gamma}} = 1 - \frac{1}{[p_0, p_1; p_2, p_3]}.
\end{aligned}$$

(b) We have

$$(2, 4, 3) = (0, 2, 1) + 2(1, 1, 1) \text{ and } (1, -1, 0) = -(0, 2, 1) + (1, 1, 1).$$

This implies that $p_0, p_1, p_2, p_3 \in \mathbb{R}P^2$ lie on a Line and

$$[p_0, p_1; p_2, p_3] = \frac{2}{1} \Big/ \frac{1}{-1} = -2.$$

(c) Note first that the line l through two different points $v_0, v_1 \in \mathbb{R}^3$ is given by

$$l = \{\lambda v_0 + (1-\lambda)v_1 \mid \lambda \in \mathbb{R}\}.$$

Moreover, the point $\lambda v_0 + (1-\lambda)v_1$ divides the line segment joining v_0, v_1 in the ratio $(1-\lambda)/\lambda$. Therefore, if $A = v_0, B = v_1, C = \lambda v_0 + (1-\lambda)v_1, D = \mu v_0 + (1-\mu)v_1$ are four different points on a line $l \subset \mathbb{R}^3$, then the ratios AC/CB and AD/DB are given by

$$\frac{AC}{CB} = \frac{1-\lambda}{\lambda}, \quad \frac{AD}{DB} = \frac{1-\mu}{\mu}.$$

Now, the cross ratio of $A = [v_0], B = [v_1], C = [v_2], D = [v_3] \in \mathbb{R}P^2$ is given by

$$[A, B; C, D] = \frac{1-\lambda}{\lambda} \Big/ \frac{1-\mu}{\mu} = \frac{AC}{CB} \Big/ \frac{AD}{DB}.$$

Exercise 22. One easily checks that $\det A = \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = -1$, and therefore A has a normal form $B^\top AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ for some $B \in GL(3, \mathbb{R}$, by the classification result on page 36 of the script. Consequently, C is non-empty and non-singular. The equation for the pole $[x] = [x_1, x_2, x_3]$ is

$$(x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ 4v_3 \\ v_3 \end{pmatrix},$$

since the Points in l are given as $[v_1, 4v_3, v_3] \in \mathbb{R}P^2$. Evaluating this equation in the special cases $v = [1, 0, 0] \in l$ and $v = [0, 4, 1]$, we obtain

$$(x_1, x_2, x_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (x_1, x_2, x_3) \begin{pmatrix} 0 \\ 6 \\ 11 \end{pmatrix} = 0$$

yields $x_1 = 0$, $x_2 = -11$, $x_3 = 6$, i.e., the pole is $[x] = [0, -11, 6] \in \mathbb{R}P^2$.