Geometry III/IV

Solutions, set 9.

Exercise 23. A matrix corresponding to the conic $C \subset \mathbb{R}P^2$ is

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Since $\det A = -1$, the conic is non-singular. Moreover, we have $P = [1, 0, 0] \in C$, so C is non-empty.

(a) Since $P \in C$, the polar Line l_0 of P with respect to C is actually the tangent Line to C in P. It is given by the equation

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0.$$

This leads to $l_0 = \{[y_1, y_2, y_3] \mid y_3 = 0\}$.

(b) The pole Q of l with respect to C is given by the equation

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2t \\ s \\ t \end{pmatrix} = 0$$

for all $s, t \in \mathbb{R}$. This reduces to $x_2 = 0, x_1 = -2x_3$, i.e., $Q = [-2, 0, 1]$.

(c) Normalized eigenvectors of A are

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

with corresponding eigenvalues $1, 1, -1$. Choosing the orthogonal matrix

$$B = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix},$$

we conclude that

$$B^\top A B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Defining

$$f([x]) = [B^{-1}x]$$

we conclude that

$$\begin{aligned} f(C) = \{[B^{-1}x] \mid x^\top A x = 0\} &= \{[y] \mid y^\top B^\top A B y = 0\} \\ &= \{[y] \mid {y_1}^2 + {y_2}^2 - {y_3}^2 = 0\}. \end{aligned}$$

Exercise 24.

(a) We have

$$\begin{aligned} l_A &= \{[x, y, z] \mid x = 0\}, & l_B &= \{[x, y, z] \mid y = 0\}, \\ l_C &= \{[x, y, z] \mid z = 0\}, & l_D &= \{[x, y, z] \mid x + y + z = 0\}. \end{aligned}$$

(b) We have

$$\begin{aligned} AB &= \{[x, y, z] \mid z = 0\}, & CA &= \{[x, y, z] \mid y = 0\}, \\ AD &= \{[x, y, z] \mid y = z\}, & BC &= \{[x, y, z] \mid x = 0\}, \\ BD &= \{[x, y, z] \mid x = z\}, & CD &= \{[x, y, z] \mid x = y\}. \end{aligned}$$

This implies that $A' = [0, 1, 1]$, $B' = [1, 0, 1]$, $C' = [1, 1, 0]$, and

$$\begin{aligned} A'B' &= \{[x, y, z] \mid z - (x + y) = 0\}, \\ C'A' &= \{[x, y, z] \mid y - (x + z) = 0\}, \\ B'C' &= \{[x, y, z] \mid x - (y + z) = 0\}. \end{aligned}$$

Consequently, we get

$$X = BC \cap B'C' = [0,1,-1], \quad Y = CA \cap C'A' = [1,0,-1],$$
$$Z = AB \cap A'B' = [1,-1,0].$$

Collinearity of X, Y, Z follows from the fact that

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(c) The dual statement reads as follows: Let l_A, l_B, l_C, l_D be given as in (a). Let $l_{A'}$ be the Line through the intersection Points $l_A \cap l_D$ and $l_B \cap l_C$. Let $l_{B'}$ be the Line through the intersection Points $l_B \cap l_D$ and $l_C \cap l_A$. Let $l_{C'}$ be the Line through the intersection Points $l_C \cap l_D$ and $l_A \cap l_B$. Let l_X be the Line through the intersection Points $l_B \cap l_C$ and $l_{B'} \cap l_{C'}$. Let l_Y be the Line through the intersection Points $l_C \cap l_A$ and $l_{C'} \cap l_{A'}$. Let l_Z be the Line through the intersection Points $l_A \cap l_B$ and $l_{A'} \cap l_{B'}$. Then the three Lines l_X, l_Y, l_Z meet in a common Point.

Exercise 25.

(a) Note that $x^\top A y = \langle Ax, y \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in $\mathbb{R}^3$. Thus we have

$$l_P = \{[y] \mid \langle Ax, y \rangle = 0\}.$$

But then the dual of l_P is just $[Ax]$, since Ax is a non-zero vector orthogonal to the plane $E \subset \mathbb{R}^3$ through the origin determined by the projective line l_P. We conclude that

$$\begin{aligned} C^\top &= \{[Ax] \mid x \in C\} = \{[Ax] \mid x^\top A x = 0\} \\ &= \{[y] \mid y^\top (A^{-1})^\top A A^{-1} y = 0\} = \{[y] \mid y^\top A^{-1} y = 0\}, \end{aligned}$$

by using the fact $(A^{-1})^\top = (A^\top)^{-1} = A^{-1}$. A^{-1} is non-singular since A is non-singular.

(b) The dual statement to the Theorem of Pascal reads as follows:

Let C be a non-singular conic and l_1, l'_1, l_2, l'_2 and l_3, l'_3 be six distinct tangents to C. Let r_3 be the Line passing through the intersection Points $l_1 \cap l'_2$ and $l_2 \cap l'_1$. Let r_2 be the Line passing through the intersection Points $l_1 \cap l'_3$ and $l_3 \cap l'_1$. Let r_1 be the Line passing through the intersection Points $l_2 \cap l'_3$ and $l_3 \cap l'_2$. Then the three Lines r_1, r_2, r_3 are concurrent.