Lecture 5

In the last lecture we introduced quantifiers and discussed negations of statements containing quantifiers. We also introduced arbitrary unions and intersections of sets using indices. Today's lecture is dedicated to Proof Techniques. We will focus on two particular Proof Techniques, namely **Indirect Proof** *and* **Induction**.

Before we discuss the principle of Indirect Proof, let us start with a bit of logic:

Here is a very important pitfall: The negation of "If A then B" is not "If A then (not B)". Let us find out what the negation is:

A	B	not B	If A then B	If A then (not B)	not(If A then B)
False	False	True	True	True	False
False	True	False	True	True	False
True	False	True	False	True	True
True	True	False	True	False	False

This shows that the correct negation is "A and (not B)". This plays an important role for some Indirect Proofs.

Indirect Proof: Let A be the statement we like to prove. In the Indirect Proof, we assume that the negation of A is true and show that this assumption leads to a contradiction. This shows that "not A" is false, and therefore A must be true.

A classical example of an Indirect Proof is Euclid's proof that there are infinitely many prime numbers. We assume the opposite, namely, that there are only finite many prime numbers $p_1, \dots, p_n$. Then we conclude that the number $N := p_1 p_2 \cdots p_n + 1$ must contain a prime number different from $p_1, \dots, p_n$. This is a contradiction. Therefore we must have infinitely many prime numbers.

Example of an Indirect Proof:

Theorem. *The sequence $x_1, x_2, x_3, \dots,$ given by*

$$x_{n+1} = x_n^2 + 1 \qquad \forall \ \ n \geq 1$$

does not have a limit for any **real** *initial value $x_1 \in \mathbb{R}$.*

Indirect Proof: Assume that the theorem is wrong. Then there exists a real initial value $x_1 \in \mathbb{R}$ such that the sequence $x_1, x_2, \dots$ has a limit. Let $z = \lim_{n\to\infty} x_n$ be this limit. The recursion formula guarantees that all elements x_n are real, therefore also the limit z must be real. Using the recursion formula, we conclude that

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n^2 + 1 = z^2 + 1.$$

This shows that $z^2 - z + 1 = 0$, i.e.,

$$z = \frac{1 \pm \sqrt{3}}{2} \notin \mathbb{R}.$$

This is a contradiction. □

Another important proof technique is Induction.

Induction: Given a sequence of open statements $A(n)$, indexed by integers $n \geq n_0$. The idea of proof goes as follows:

- **Start of Induction:** Show that $A(n_0)$ is true.
- **Induction Step:** Show that if $A(n)$ is true for an arbitrary integer $n \geq n_0$, then $A(n+1)$ is also true.

Conclusion: $A(n)$ is true for all $n \geq n_0$.

Note that induction provides the following:

- $A(1)$ is true by the Start of Induction.
- Applying the Induction Step to $A(1)$, we see that $A(2)$ is true.
- Applying the Induction Step to $A(2)$, we see that $A(3)$ is true.
- Applying the Induction Step to $A(3)$, we see that $A(4)$ is true.

$\vdots$

Example: We prove the following statement by Induction.

Theorem. *Let $q \neq 1$ and $n \in \mathbb{N}$. Then we have*

$$1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}. \tag{1}$$

Note that the sum in the theorem is called a **geometric series**. The statement $A(n)$ is here that equation (1) holds for a fixed integer n.

Induction Proof:

Start of Induction (n=1): We obviously have

$$1 = \frac{q-1}{q-1},$$

i.e., (1) holds for $n = 1$.

Induction Step: Assume (1) holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} 1 + q + q^2 + \cdots + q^{n-1} + q^n &= \frac{q^n - 1}{q - 1} + q^n \\ &= \frac{q^n - 1 + q^n(q-1)}{q-1} = \frac{q^{n+1} + q^n - q^n - 1}{q-1} = \frac{q^{n+1} - 1}{q-1}. \end{aligned}$$

This shows that (1) also holds for $n+1$, finishing the Induction Step. $\square$

Sometimes the Induction Step is more involved. The following variant works just as well and is called **Strong Induction**:

- **Start of Induction:** Show that $A(1), A(2), \ldots, A(k)$ is true.
- **Induction Step:** Let $n \geq k+1$ be an arbitrary integer. Show that if $A(j)$ is true for all integers $1 \leq j \leq n$, then $A(n+1)$ is also true.

Conclusion: $A(n)$ is true for all $n \in \mathbb{N}$.

Example: We prove the following statement by Strong Induction.

Let $x_1, x_2, \ldots$ be a sequence with $x_1 = 1$, $x_2 = 3$ and

$$x_n = x_{n-1} + x_{n-2} \quad \forall \;\; n \geq 3.$$

Then we have

$$x_n < \left(\frac{7}{4}\right)^n \tag{2}$$

for all integers $n \in \mathbb{N}$.

The statement $A(n)$ is here that equation (2) holds for a fixed $n \in \mathbb{N}$.

Strong Induction Proof:

Start of Induction (n=1,2): We obviously have

$$\begin{aligned} x_1 &= 1 < \frac{7}{4}, \\ x_2 &= 3 < \left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16}. \end{aligned}$$

This shows that $A(1)$ and $A(2)$ are true.

Induction Step: Let $n \geq 2$. Assume that $A(j)$ is true for all $1 \leq j \leq n$. Then we have

$$\begin{aligned} x_{n+1} &= x_n + x_{n-1} < \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} \\ &= \left(\frac{7}{4}\right)^{n-1} \left(1 + \frac{7}{4}\right) \\ &< \left(\frac{7}{4}\right)^{n-1} \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{n+1}, \end{aligned}$$

since

$$\left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16} > 1 + \frac{7}{4} = 2\frac{3}{4}.$$

This shows that $A(n+1)$ is also true, finishing the Induction Step. $\square$