Lecture 7

In the last lecture we discussed another Proof Technique, the **Contrapositive Method**. *We also discussed the meaning of* **necessary** *and* **sufficient conditions**. *In this lecture we will consider* **modelling problems**.

Modelling problems are real world problems which need to be translated into a mathematical problem in order to be "solved". There are usually three stages associated to a modelling problem:

(a) **Modelling stage:** We first translate a real world problem into a mathematical problem, that is, we try to identify knowns and unknowns, name them and identify mathematical relations between them. This leads to a well-posed mathematical problem (which might be an idealisation/approximation of the real world probelm).

(b) **Solution stage:** Once the mathematical problem is derived, we try to find a solution for it.

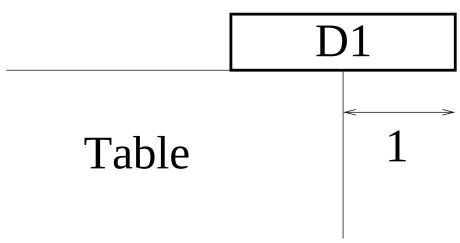
(c) **Looking back stage:** Finally, we investigate the relevance of the solution for the original real world problem. Does the solution make sense? Does it really solve the original problem? If there are doubts, we may have to go back to stage (a) and construct a new mathematical model.

The best way to illustrate this is to look at a concrete example:

Example (Sharp 1954, Overhanging domino problem): Given n identical domino blocks. Aim: Create the largest possible overhang by stacking them up over the table's end, subject to the laws of gravity.

(a) **Modelling stage:**

We assume that all dominos have length 2. We start with one domino, denoted by D_1, and we can obviously place it on the table that it sticks out 1. The the centre of gravity is precisely over the edge of the table and we assume that this is sufficient that it does not tip over.

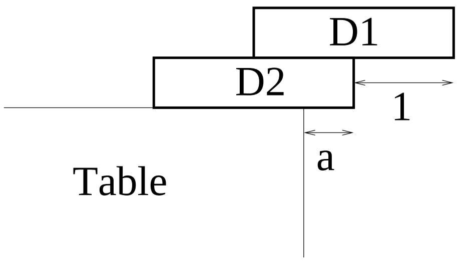


The problem is that we cannot place a second domino on top of it sticking out further, since then the centre of gravity of both dominos would lie to the right of the table and the dominos would tip over.

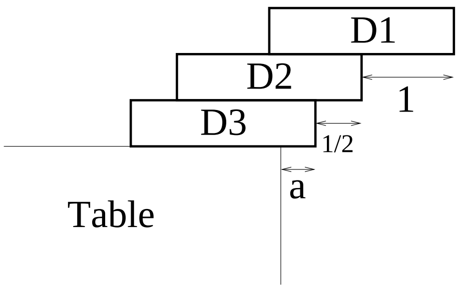
This suggests that the answer is: *The overhang cannot be larger than 1!*

But this seems to easy!!!

We wonder whether it would be helpful **not** to move the domino sitting on the table to its maximal position and therefore to have some freedom to move another domino on top of it a bit further to the right. After some thoughts we may realise that we should change our approach: replace the "bottom up" approach by a "top bottom" approach. We may use our first observation about one domino to place it on top of another one, denoted by D_2, so that D_1 sticks out 1 above D_2. Then D_1 will not tip over on top of D_2. But we must also make sure that the common centre of gravity of D_1 and D_2 lies precisely over the edge of the table. We assume D_2 sticks out $a > 0$ relative to the right end of the table.



The total centre of gravity of $D_1 \cup D_2$ is $\frac{1}{2}(a - 1 + a) = \frac{2a-1}{2}$ relative to the right end of the table. This leads to $a = 1/2$. Now we know how we must deal with a third domino D_3.



We know already that D_1 and D_2 sit safely on top of D_3, and the total centre of gravity of $D_1 \cup D_2 \cup D_3$ is $\frac{1}{3}(a-1+a+1/2-1+a+1/2) = \frac{3a-1}{3}$ relative to the right end of the table. This leads to $a = 1/3$. Here emerges a pattern, and we courageously formulate the following conjecture:

Conjecture. *Let k dominos $D_k, D_{k-1}, \ldots, D_1$ be stacked on a table, each one on top of the previous one. Assume that D_j sticks out to the right of D_{j-1} by $1/j$, and D_k sticks out to the right of the table by $1/k$. Then the centre of gravity of $D_1 \cup \cdots \cup D_k$ lies right above the right end of the table.*

It this conjecture were true for all k, we would have a safe configuration of n dominos sticking out in total by $1 + 1/2 + \cdots + 1/n$ over the table. We would need to think a bit more to make sure that this value is maximal. But the completely suprising fact is that – having an unlimited amount of dominos – we can create a stack of dominos as far to the right as we wish, since the harmonic series is divergent.

(b) **Solution stage:** We like to prove the conjecture. The natural choice of proof is **Induction**.

Induction Start (k=1): The conjecture is trivially true in this case.

Induction Step: Assume the conjecture is true for some $k \geq 1$. We stack $D_1, \ldots, D_k$ on top of D_{k+1} with the required overhangs, and assume that D_{k+1} sticks out $\frac{1}{k+1}$ to the right of the table. The induction hypothesis tells us that the total centre of gravity of $D_1 \cup \cdots \cup D_k$ lies precisely over the right end of D_{k+1}, i.e., $\frac{1}{k+1}$ relative to the right end of the table. To calculate the centre of gravity of $D_1 \cup \cdots \cup D_{k+1}$ relative to the right end of the table, we need to calculate the weighted average

$$\frac{1}{k+1}\left(1 \cdot (\frac{1}{k+1} - 1) + k \cdot \frac{1}{k+1}\right) = 0,$$

which means that the centre of gravity of $D_1 \cup \cdots \cup D_{k+1}$ lies precisely over the right end of the table. This shows that the statement holds also true for $k+1$.

(c) **Looking back stage:** Once having obtained this result, we may try it out with concrete objects like identical books etc., and see whether our theoretical result agrees with practical experiments. The very slow divergence of the harmonic series means that, in practice, we don't really encounter a too big overhang. But already the fact that a book might be placed into a position that even its left end is beyond the table's right end is quite impressive.