Lecture 8

In the last lecture we discussed **modelling problems**, *and in this lecture we will discuss an important concept to compare the size of sets, in particular if they are infinite. We start to introduce the* **cardinality** *of a set and will then present ideas introduced by* Georg Cantor (1845-1918) *to compare finite and infinite sets.*

It is often important to talk about the "number of elements of a set". For finite sets, we can compare them via the "number of their elements". We give this concept a proper mathematical name:

Definition. *Let X be a set. Then the number of elements in X is called* **cardinality** *of X. We write $|X|$ for the cardinality of X. If a set X has infinitely many elements, we write $|X| = \infty$.*

Examples: We have $|\{a, b, c\}| = 3$, $|\{p \in \mathbb{N} \mid p \text{ is a prime}\} = \infty$, $|\{0, 1, 2, \ldots, k\}| = k + 1$.

Now we want to introduce concepts to compare not only finite sets but also infinite sets. Here are the crucial definitions, based on the notions of injectivity and bijectivity.

Definition. *Let X and Y be two sets. We say that X* has cardinality smaller than or equal to the cardinality of Y *if there exists an injective map $f : X \to Y$. We then write $|X| \leq |Y|$. We say that X* and Y have the same cardinality *if there exists a bijective map $f : X \to Y$; we then write $|X| = |Y|$. The cardinality of X is strictly smaller than the cardinality of Y if there exists an injective map $f : X \to Y$ but there is no bijective map $g : X \to Y$. We then write $|X| < |Y|$.*

For **finite** sets, we have the following obvious property: If $X \subset Y$ and both sets have the same cardinality then $X = Y$. This is **no longer true for infinite sets**. Strange phenomena can occur in the case of infinite sets, as discussed in the following example.

Example: Let $X = (0, 1)$ and $Y = (1, \infty)$. Since X has obviously the same cardinality as $(k, k + 1)$ for $k \in \mathbb{N}$, it seems that

$$Y = (1, \infty) \supset \bigcup_{k=1}^{\infty} (k, k + 1)$$

must have a much larger cardinality than X. On the other hand, we have an bijective map $f : X \to Y$ by $f(x) = 1/x$, which shows that X and Y have the same cardinality.

Here is another example:

Example: The sets $\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integers) have the same cardinality. We use the bijection

$$f : \mathbb{N} \to \mathbb{Z}, \quad f(k) = \begin{cases} l & \text{if } k = 2l, \\ -l & \text{if } k = 2l+1. \end{cases}$$

There is another problem with the above definition: We would like to have that if $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$. The inequalities $|X| \le |Y|$ and $|Y| \le |X|$ only mean that there are injective maps $f_1 : X \to Y$ and $f_2 : Y \to X$. To conclude that $|X| = |Y|$, we would need to show that there is a bijective map $g : X \to Y$. This is by no means a trivial problem. In fact, it is true, but it is a theorem called the **Cantor-Bernstein-Schroeder Theorem**.

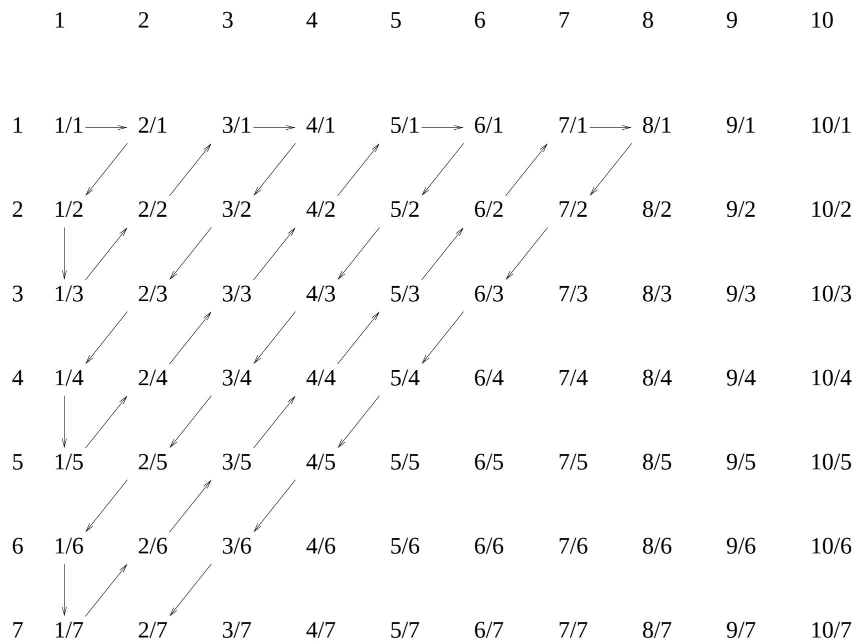
Now, we consider the following natural question:

Question: Which of the two subsets $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ is larger? Are there more irrational numbers than rational numbers?

Infinite sets which have the same cardinality as the set of natural numbers $\mathbb{N}$ are called **countable sets**. An infinite set which is not countable is called **uncountable**. For every countable set X, we can *enumerate* its elements, i.e., we can write

$$X = \{x_1, x_2, x_3, \dots\}.$$

We have already seen that $\mathbb{Z}$ is countable. But the seemingly much larger set $\mathbb{Q}$ is also countable. We prove this first for the set of all positive rational numbers. We produce an infinite square scheme of numerators (upper horizontal line) and denominators (left vertical column) and go through them one by one in the diagonal fashion shown here, omitting a rational if it was already chosen earlier:



We end up with the enumeration

$$\{1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7, 8, \frac{7}{2}, \frac{5}{4}, \frac{4}{5}, \frac{2}{7}, \dots\},$$

and one easily checks that all positive rationals are chosen during this procedure. Once we have enumerated all positive rationals by $\{q_1, q_2, q_3, q_4, \dots\}$, we can show the countability of the set of all rationals $\mathbb{Q}$ by ordering them as

$$\mathbb{Q} = \{0, q_1, -q_1, q_2, -q_2, q_3, -q_3, \dots, \}.$$

Next, we show that the set of all real numbers in the interval $[0, 1]$ is uncountable:

Indirect Proof: Assume the set of all real numbers in $[0, 1]$ is countable. Then we can write

$$[0, 1] = \{x_1, x_2, x_3, \dots\}.$$

Write the numbers $x_1, x_2, x_3, \dots$ in their decimal representation as a list:

$$\begin{aligned}
x_1 &= 0.x_{11}x_{12}x_{13}x_{14}\dots, \\
x_2 &= 0.x_{21}x_{22}x_{23}x_{24}\dots, \\
x_3 &= 0.x_{31}x_{32}x_{33}x_{34}\dots, \\
&\vdots
\end{aligned}$$

Now we construct a number $y = 0.y_1y_2y_3 \cdots \in [0,1]$ as follows: For all $i \in \mathbb{N}$, we choose

$$y_i = \begin{cases} 5, & \text{if } x_{ii} \neq 5, \\ 6, & \text{if } x_{ii} = 5. \end{cases}$$

Then y is obviously not in the list. But this contradicts to the assumption that $x_1, x_2, x_3 \ldots$ are **all** numbers in the interval $[0,1]$. $\square$

With the above arguments, you should be now able to prove that the union of two countable sets is, again, countable. If all the irrational numbers would be countable, then the real numbers $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \backslash \mathbb{Q})$ would then also be countable. But the subset $[0,1] \subset \mathbb{R}$ is already uncountable. Therefore, we conclude that there are many more irrational numbers than rational numbers.

Finally, let us show that the cardinality of a power set $\mathcal{P}(X)$ is always strictly larger than the cardinality of the set X. This is obviously true for finite sets, since then we have (see Question 4 in Set Problems (Week 2))

$$|\mathcal{P}(X)| = 2^{|X|}.$$

For an infinite set X, we have the trivial injective map

$$f : X \to \mathcal{P}(X), \quad f(x) = \{x\} \subset X,$$

i.e., $|X| \leq |Y|$. It remains to rule out that there is a bijective map $g : X \to \mathcal{P}(X)$. Assume that such a bijective map g exists. Then we have a $1-1$ correspondence between elements of X and subsets of X, given by

$$x \mapsto g(x).$$

Now we construct a subset $Y \subset X$ which cannot be the image of an element $x \in X$. The idea is to construct Y in such a way that it is different from every $g(x)$. We do this by establishing $x \in Y$ if $x \notin g(x)$ and $x \notin Y$ if $x \in g(x)$. Then Y differs from $g(x)$ with regards to the element x itself. The definition of Y is

$$Y = \{x \in X \mid x \notin g(x)\} \subset X.$$

The construction of Y guarantees that there is no element $x \in X$ with $Y = g(x)$, contradicting to the assumption that $g : X \to \mathcal{P}(X)$ is bijective.