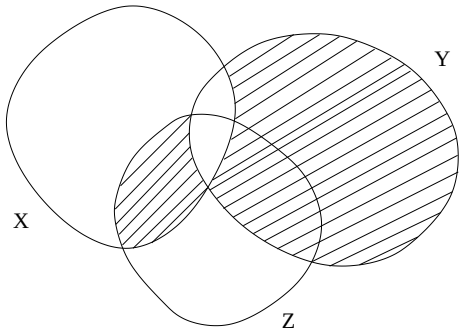
Set Problems (Week 2)

Question 1 Describe the set illustrated in the following Venn diagram.



Question 2 For two real numbers $x, y \in \mathbb{R}$, the minimum of x and y is denoted by $\min(x, y)$. Let $a, b \in \mathbb{R}$ be two real numbers. Show the following identity:

$$\{x \in \mathbb{R} \mid x \leq a\} \cap \{x \in \mathbb{R} \mid \min(x, a) \leq b\} = \{x \in \mathbb{R} \mid x \leq \min(a, b)\}.$$

Question 3 Show the following identity about the set of all positive differences of two squares:

$$\{a^2 - b^2 \mid a, b \in \mathbb{N} \cup \{0\}, a > b\} = \{2k - 1 \mid k \in \mathbb{N}\} \cup \{4j \mid j \in \mathbb{N}\}.$$

Question 4 The *power set* $\mathcal{P}(X)$ of a set X is the set of all subsets of X. For example, if $X = \{1, 2\}$ then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Another example: We have $\mathbb{N} \in \mathcal{P}(\mathbb{Q})$ and $\mathbb{Z} \in \mathcal{P}(\mathbb{Q})$. Think about the following statements and determine which of them are true, which are false. Try to explain why you think so.

1. If X is finite and has N elements, then $\mathcal{P}(X)$ is also finite and has 2^N elements.
2. If $Z = X \cap Y$, then

$$\mathcal{P}(Z) = \mathcal{P}(X) \cap \mathcal{P}(Y).$$

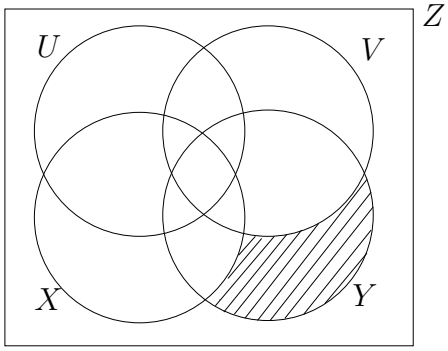
3. If $Z = X \cup Y$, then

$$\mathcal{P}(Z) = \mathcal{P}(X) \cup \mathcal{P}(Y).$$

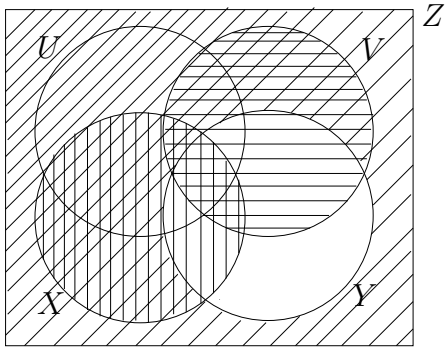
Question 5 (Jack's Dilemma) Let U, V, X, Y be four subsets of Z. Jack draws the following Venn Diagrams to show that

$$Y \cap (U^c \cap V^c \cap X^c) = (V \cup X \cup Y^c)^c, \tag{1}$$

where complements are taken with respect to the set Z.



$Y \cap (U^c \cap V^c \cap X^c)$



$V \cup X \cup Y^c$

Then he looks at the following example:

$$U = \{1\}, V = \{3, 4\}, X = \{3\}, Y = \{1, 2, 3\}$$

and $Z = \{1, 2, 3, 4\}$ and obtains

- $Y \cap (U^c \cap V^c \cap X^c) = \{2\}$,
- $(V \cup X \cup Y^c)^c = \{1, 2\}$.

While Jack's Venn Diagrams confirm (1), his example shows that (1) is not true.

Check that Jack's Venn Diagram agrees with (1) and that there is no mistake in his example. Can you explain this discrepancy?

Question 6 The *symmetric difference* of two sets X, Y is defined as

$$X \Delta Y = (X \backslash Y) \cup (Y \backslash X).$$

- Draw a Venn Diagram to illustrate $X \Delta Y$.
- Draw Venn Diagrams for $(X \Delta Y) \Delta Z$ and $X \Delta (Y \Delta Z)$.
- Show that $X \Delta Z \subset (X \Delta Y) \cup (Y \Delta Z)$.