Answers to Set Problems

Question 1 There are different ways to describe this set. One expression for it is

$$((A \cap C)\backslash B) \cup (B\backslash A).$$

Another expression is

$$(B \cup (C \cap A))\backslash(A \cap B).$$

Question 2 Let $X = \{x \in \mathbb{R} \mid x \leq a\} \cap \{x \in \mathbb{R} \mid \min(x, a) \leq b\}$ and $Y = \{x \in \mathbb{R} \mid x \leq \min(a, b)\}$. To show that the two sets X and Y are equal, we have to prove two facts. Firstly, every element of X is also an element of Y. Secondly, every element of Y is also an element of X. Here are the arguments:

- If $x \in X$ then $x \in Y$: Note that $x \leq a$ and $\min(x, a) \leq b$ implies that $x = \min(x, a) \leq b$ and, therefore $x \leq \min(a, b)$.
- If $x \in Y$ then $x \in X$: We conclude from $x \leq \min(a, b)$ that $x \leq a$, i.e., $x = \min(x, a)$, and $x \leq b$. This implies $x \leq a$ and $x = \min(x, a) \leq b$.

Question 3 As in Question 2, we have to prove firstly that every element of the left hand set is also an element of the right hand set, and secondly that every element of the right hand set is also an element of the left hand set.

Let $Y_1 := \{2k - 1 \mid k \in \mathbb{N}\}$ and $Y_2 := \{4j \mid j \in \mathbb{N}\}$. Then Y_1 is the set of all positive odd integers, and Y_2 is the set of all positive integers, divisible by 4.

Let $a, b \in \mathbb{N} \cup \{0\}$, $a > b$. Then $0 < a^2 - b^2 = (a - b)(a + b)$. If a, b are both odd or both even, the factors $a-b, a+b$ are both even and their product is therefore a positive integer, divisible by 4, i.e., $a^2 - b^2 \in Y_2$. Otherwise $a - b, a + b$ are both odd and their product is a positive odd integer, i.e., $a^2 - b^2 \in Y_1$. This shows that every element of the left hand set is also an element of the right hand set.

Now we choose an arbitrary element $2k - 1 \in Y_1$ with $k \in \mathbb{N}$. Since $k^2 - (k - 1)^2 = 2k - 1$, we see that this element is also in the left hand set. Finally, we choose an arbitrary element $4j \in Y_2$ with $j \in \mathbb{N}$. Since $(j + 1)^2 - (j - 1)^2 = 4j$, this element is also in the left hand set. This finishes the proof.

Question 4

1. The statement is true. We give names to the elements of the set X, i.e. $X = \{a_1, \dots, a_N\}$. Now, every subset of X corresponds uniquely to N yes/no choices, deciding for each of the elements a_j whether it is in the subset or not. We have 2^N possibilities to make these choices, therefore $\mathcal{P}(X)$ has exactly 2^N elements.

   (This fact can also be proved rigorously using Induction. But Induction will be introduced later, so we provided arguments avoiding this very important technique.)

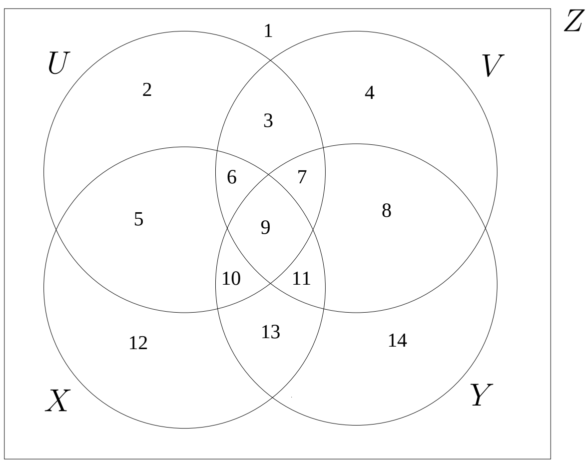
2. The statement is true.

   Let $U \in \mathcal{P}(Z)$. Then $U \subset Z$. Since $U \subset Z$ and $Z \subset X$ and $Z \subset Y$, we also have $U \subset X$ and $U \subset Y$, i.e., $U \in \mathcal{P}(X)$ and $X \in \mathcal{P}(Y)$. This shows that $U \in \mathcal{P}(X) \cap \mathcal{P}(Y)$.

   Conversely, let $U \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. Then $U \subset X$ and $U \subset Y$, i.e., $U \subset X \cap Y = Z$. This shows that $U \in \mathcal{P}(Z)$.

3. The statement is false. We only need to provide a counterexample. Let $X = \{a\}$ and $Y = \{b\}$. Then $Z = \{a, b\}$ and $Z \in \mathcal{P}(Z)$. But $Z \not\subset X$ and $Z \not\subset Y$, therefore $Z \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$.

Question 5 Both Jack's Venn Diagram and his example are correct. The problem lies in the conclusion from the Venn Diagram. We enumerate the components of both Venn Diagrams from 1 to 14 as follows:



You can easily check that these components describe the following subsets of Z:

component	subset	component	subset
1	$U^c \cap V^c \cap X^c \cap Y^c$	8	$U^c \cap V \cap X^c \cap Y$
2	$U \cap V^c \cap X^c \cap Y^c$	9	$U \cap V \cap X \cap Y$
3	$U \cap V \cap X^c \cap Y^c$	10	$U \cap V^c \cap X \cap Y$
4	$U^c \cap V \cap X^c \cap Y^c$	11	$U^c \cap V \cap X \cap Y$
5	$U \cap V^c \cap X \cap Y^c$	12	$U^c \cap V^c \cap X \cap Y^c$
6	$U \cap V \cap X \cap Y^c$	13	$U^c \cap V^c \cap X \cap Y$
7	$U \cap V \cap X^c \cap Y$	14	$U^c \cap V^c \cap X^c \cap Y$

Now, there are 16 combinations $U^* \cap V^* \cap X^* \cap Y^*$, where $*$ is either no symbol or the complement symbol, so the Venn Diagram misses out the two combinations $U^c \cap V \cap X \cap Y^c$ and $U \cap V^c \cap X^c \cap Y$. In other words, the diagram identifies the set $U \cap V^c \cap X^c \cap Y$ with the empty set (i.e., there is no region representing this set). So in the Venn Diagram the sets $Y \cap (U^c \cap V^c \cap X^c)$ and

$$T := (Y \cap (U^c \cap V^c \cap X^c)) \cup (U \cap V^c \cap X^c \cap Y) \tag{1}$$

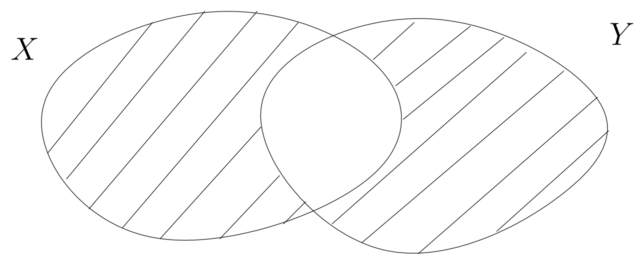
are indistiguishable, since the second set in the union (1) is represented as the empty set. Using the laws of commutativity, associativity and distributivity and, finally, De Morgan's Rule we transform the set (1) into the set $(V \cup X \cup Y^c)^c$:

$$\begin{aligned}
T &= (U^c \cap (V^c \cap X^c \cap Y)) \cup (U \cap (V^c \cap X^c \cap Y)) \\
&= (U^c \cup U) \cap (V^c \cap X^c \cap Y) \\
&= Z \cap (V^c \cap X^c \cap Y) \\
&= (V^c \cap X^c \cap Y) \\
&= (V \cup X \cup Y^c)^c.
\end{aligned}$$

Here, we see that we have to **be careful with Venn Diagrams**. While Venn Diagrams usually illustrate set relations correctly for operations on three sets, they cannot represent all 16 possibilities of intersection in the plane in the case of four sets. A remedy would be to draw Venn Diagrams with sets in $\mathbb{R}^3$, but this would be hard to imagine.

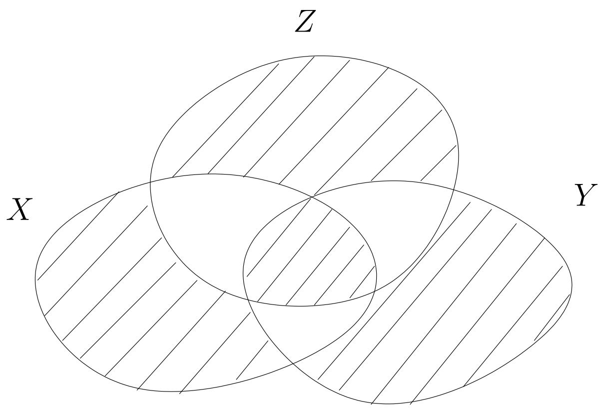
Question 6

- The Venn Diagram looks as follows:



$X \Delta Y$

- The Venn Diagram for both sets looks the same:



$(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$

  Both sets are equal and can be described in words as follows: They consist of all elements which belong to only one of the three sets or lie in the intersection of all three sets X, Y, Z. Therefore, another way of describing these sets is

$$(X \cap Y^c \cap Z^c) \cup (X^c \cap Y \cap Z) \cup (X \cap Y \cap Z^c) \cup (X \cap Y \cap Z).$$

- Let $x \in X \Delta Z$. Then x belongs to precisely one of the two sets X and Z. Now we have two cases to consider: The first case is $x \in Y$ and the second case is $x \notin Y$. One of these two cases is always fulfilled.

  Firstly, assume that $x \in Y$. Since x belongs to precisely one of the two sets X and Z, it does not belong to either X or to Z. If x does not belong to X, then $x \in X \Delta Y$. If x does not belong to Z, then

$x \in Y \Delta Z$. So we conclude in the first case that we always have $x \in (X \Delta Y) \cup (Y \Delta Z)$.

Secondly, assume that $x \notin Y$. Since x belongs to precisely one of the two sets X and Z, it belongs to either X or to Z. If x belongs to X, then $x \in X \Delta Y$. If x belongs to Z, then $x \in Y \Delta Z$. So we conclude in the second case that we always have $x \in (X \Delta Y) \cup (Y \Delta Z)$.

This shows that

$$x \in X \Delta Y \quad \Rightarrow \quad x \in (X \Delta Y) \cup (Y \Delta Z),$$

finishing the proof of the inclusion.