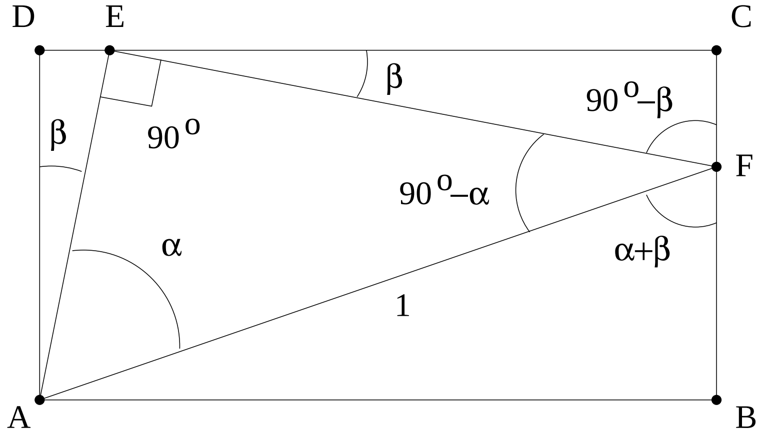
Answers to Writing Maths Problems

Question 1 (a) We have the following angles:



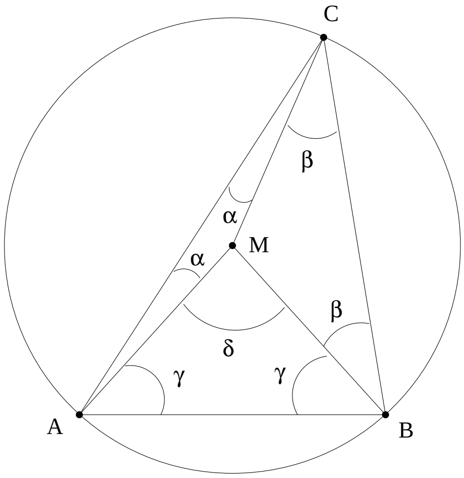
We derive the following lengths:

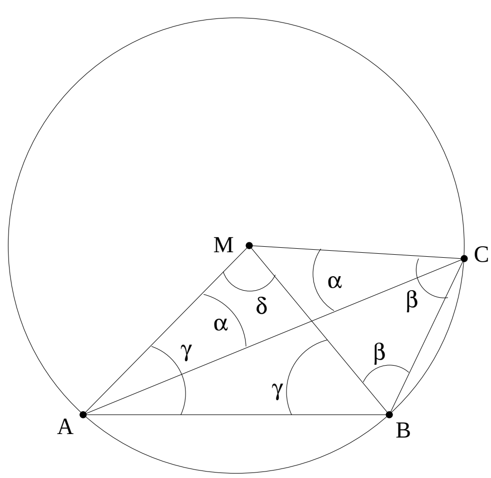
$$
\begin{aligned}
|\overline{AE}| &= \cos\alpha, & |\overline{EF}| &= \sin\alpha, \\
|\overline{AD}| &= \cos\alpha\cos\beta, & |\overline{DE}| &= \cos\alpha\sin\beta, \\
|\overline{EC}| &= \sin\alpha\cos\beta, & |\overline{CF}| &= \sin\alpha\sin\beta, \\
|\overline{AB}| &= \cos(\alpha+\beta), & |\overline{BF}| &= \sin(\alpha+\beta).
\end{aligned}
$$

(b) Since opposite sides of a rectangle have the same length, we obtain the *addition formulas for the trigonometric functions*:

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta, \\
\sin(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.
\end{aligned}
$$

Question 2 Geometrically, three cases have to be distinguished: The first case is if M lies on one of the sides of the triangle ΔABC (here we have two cases, M lying on the side AC or on the side BC, but these cases are symmetric counterparts of each other), the second case is if M lies inside the triangle ΔABC, and the third case is if M lies outside the triangle ΔABC (here we have again two cases, M lying to the right of the triangle or to the left, but these cases are symmetric counterparts of each other). Useful additional lines are the segments $\overline{MA}, \overline{MB}$ and $\overline{MC}$. Here are the illustrations of cases 2 and 3:

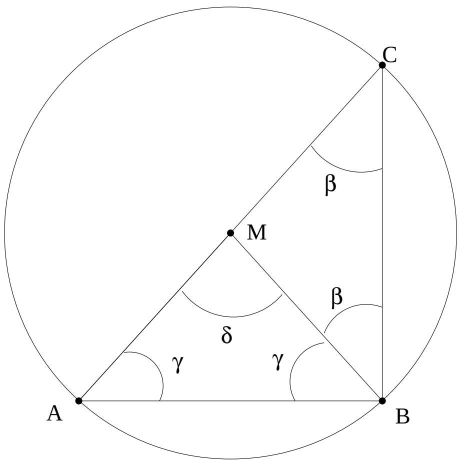




Proof: Let $r > 0$ be the radius of the circle. Then we have

$$r = |\overline{MA}| = |\overline{MB}| = |\overline{MC}|. \tag{1}$$

Let us first consider the first case, assuming without loss of generality that $M \in \overline{AC}$, see the following illustration:



We conclude from (1) that the triangles ΔAMB and ΔBMC are isosceles and we have equal base angles (by Fact (2)), i.e.,

$$\begin{aligned} \gamma &= \measuredangle MAB = \measuredangle MBA, \\ \beta &= \measuredangle MBC = \measuredangle MCB. \end{aligned}$$

Since the sum of angles in the triangle ΔABC is 180° (see Fact (1)), we conclude in the first case that

$$2\gamma = 180^\circ - 2\beta. \tag{2}$$

Applying the same fact to the triangle ΔAMB, we obtain

$$2\gamma = 180^\circ - \delta. \tag{3}$$

Combining (2) and (3), we conclude

$$\measuredangle AMB = \delta = 2\beta = 2\measuredangle ACB,$$

finishing the proof in this geometric case.

Now let us use similar arguments for the cases 2 and 3 simultaneously: We conclude from (1) that the triangles ΔAMB, ΔBMC and ΔCMA are isosceles and we have equal base angles (by Fact (2)), i.e.,

$$\begin{aligned} \gamma &= \measuredangle MAB = \measuredangle MBA, \\ \beta &= \measuredangle MBC = \measuredangle MCB, \\ \alpha &= \measuredangle MAC = \measuredangle MCA, \end{aligned}$$

Since the sum of angles in the triangle ΔABC is 180° (see Fact (1)), we conclude in the second case that

$$2\gamma = 180^\circ - 2\alpha - 2\beta, \tag{4}$$

and in the third case

$$2\gamma = 180^\circ + 2\alpha - 2\beta. \tag{5}$$

Applying the same fact to the triangle ΔAMB, we obtain in both cases (3) above, again. Combining (4) and (3) in the second case and (5) and (3) in the third case, we conclude in the second case that

$$\measuredangle AMB = \delta = 2(\alpha + \beta) = 2\measuredangle ACB,$$

and in the third case

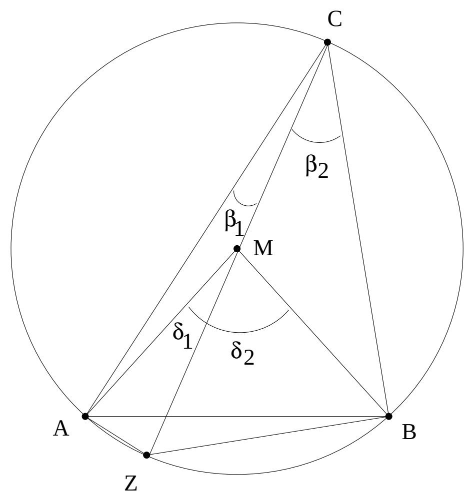
$$\measuredangle AMB = \delta = 2(\alpha - \beta) = 2\measuredangle ACB.$$

This shows the identity

$$\measuredangle AMB = 2\measuredangle ACB$$

in the remaining two geometric cases, finishing the proof.

Remark: Interestingly, it turns out that the very special first case, introduced above, can be used to derive the result also for cases 2 and 3. Here is a short explanation how the proof of the first case can be used to give a proof of case 2: The idea is to introduce the additional point Z as the second intersection point of the line CM with the circle and to look at the triangles ΔAZC and ΔZCB.



The triangles ΔAZC and ΔZCB represent Case 1 and we can conclude from the above result for case 1 that $\delta_1 = 2\beta_1$ and $\delta_2 = 2\beta_2$. Combining these results shows for the triangle ΔABC:

$$\measuredangle AMB = \delta_1 + \delta_2 = 2(\beta_1 + \beta_2) = 2\measuredangle ACB.$$

Similar arguments derive the result for case 3 from case 1.

Question 3 *This is the original text:*

Definition. *Let $q \in \mathbb{R}$. We call*

$$S_n(q) = 1 + q + q^2 + \cdots + q^{n-1} \qquad (*)$$

the geometric series of q of length n.

Theorem. *Let $q \neq 1$. Then we have*

$$S_n(q) = \frac{q^n - 1}{q - 1}. \qquad (\Box)$$

Proof. Multiplication of $(*)$ with q gives

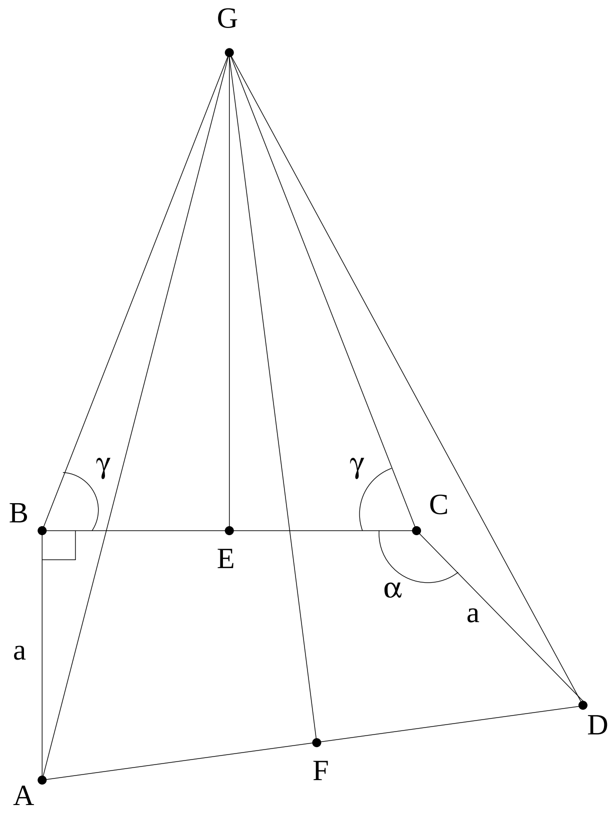
$$qS_n(q) = q + q^2 + q^3 + \cdots + q^n. \qquad (\Diamond)$$

Subtracting $(*)$ from $(\Diamond)$, and observing that most terms cancel out, leads to

$$qS_n(q) - S_n(q) = (q-1)S_n(q) = q^n - 1.$$

Now, division by $(q-1) \neq 0$ yields $(\Box)$, finishing the proof. $\Box$

Question 4 This is a classical case where a sketch is misleading and **does not represent a really occuring situation**. In fact, constructing the diagram more carefully leads to the following configuration:



The same arguments show that the triangles ΔABG and ΔDCG are congruent and that, therefore $\gamma + 90° = \measuredangle ABG = \measuredangle DCG$. Moreover, we also have $\gamma = \measuredangle EBG = \measuredangle ECG$, but this only tells us that

$$\alpha + 2\gamma + 90° = 360°,$$

i.e., $\alpha = 270° - 2\gamma > 90°$ since $\gamma < 90°$, which is no contradiction at all.

Question 5 *This is the original text:*

We first introduce the notions of r-separating and r-covering sets.

Definition 1. *Let $A \subset \mathbb{R}$ be a subset and $r > 0$. A finite set $S := \{x_1, \ldots, x_n\} \subset A$ is called* r-separating, *if the open intervals $(x_i - r, x_i + r)$ are pairwise disjoint.*

An finite r-separating set $S = \{x_1, \ldots, x_n\} \subset A$ is called maximally r-separating, *if any strictly bigger set $S' \subset A$ of finitely many points is no longer r-separating.*

Example. *Let A be the closed interval $[0, 10]$. Then the set $S := \{0, 2, 4, 6, 10\} \subset A$ is 1-separating, but not maximally 1-separating, since the bigger set $S' := \{0, 2, 4, 6, 8, 10\} \subset A$ is also 1-separating.*

Definition 2. *Let $A \subset \mathbb{R}$ be a subset and $r > 0$. A finite set $S := \{x_1, \ldots, x_n\} \subset A$ is called* r-covering, *if the union of the open intervals $(x_i - r, x_i + r)$ covers the set A.*

Example. *Let $B := \{1/n \mid n \in \mathbb{N}\}$. Then the finite set $S := \{1, 1/2, 1/4, 1/8\} \subset B$ is $1/8$-covering.*

Now we present the main result of this note.

Theorem. *Let $A \subset \mathbb{R}$ be a subset and $r > 0$. If the finite set $S \subset A$ is a maximally r-separating set, then S is also a $2r$-covering set.*

Proof. Let the finite set $S \subset A$ be given by $\{x_1, \ldots, x_n\}$. Assume S would not be $2r$-covering. Then we could find a point $x \in A$ which is not in the union of the intervals $(x_i - 2r, x_i + 2r)$. This would mean that x has distance greater or equal to $2r$ to all the points x_i. Therefore, the strictly bigger set $S' := \{x_1, \ldots, x_n, x\} \subset A$ would also be r-separating. This is a contradiction to the assumption that S is **maximally** r-separating. □