More Logic and Sets Problems (Week 4)

Question 1 For each of the following statements, restate it in English without using the abbreviating quantifiers. Try to explain in your own words what it means. Finally, write down its negation.

1. Let $(x_n)_{n=1}^{\infty}$ be a given sequence of real numbers. The first mathematical statement reads as follows:

$$\forall C > 0 \;\; \exists n \in \mathbb{N} : \quad x_n > C.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The second mathematical statement reads as follows:

$$\exists L > 0 \;\; \forall x, y \in \mathbb{R} : \quad |f(x) - f(y)| \leq L|x - y|.$$

3. Let X, Y be two sets and $g : X \to Y$ be a given map. The third mathematical statement reads as follows:

$$\forall y \in Y \;\; \exists x \in X : \quad y = g(x).$$

Question 2 The following infinite intersections and unions represent relatively simple subsets of $\mathbb{R}$. Find these simple sets in each case and justify your answer.

1. $\displaystyle\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 1\right)$

2. $\displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{2}{n}\right)$

3. $\displaystyle\bigcup_{n\in\mathbb{N}} [1, n)$

4. $\displaystyle\bigcup_{q\in\mathbb{Q}} \left(q - \frac{1}{1000}, q + \frac{1}{1000}\right)$

Question 3 Recall that the *power set* $\mathcal{P}(X)$ of a set X is the set of all subsets of X. Solve the following "power set problems".

(i) Let $X = \{1, 2, 3, 4, 5\}$ and

$$\begin{aligned} R &= \{Z \in \mathcal{P}(X) \mid 1 \in Z \text{ and } 5 \notin Z\}, \\ S &= \{Z \in \mathcal{P}(X) \mid Z \text{ has three elements}\}. \end{aligned}$$

Determine the set $R \cap S$.

(ii) Decide whether the following equality holds or not:

$$\mathcal{P}(\mathbb{N}) = \bigcup_{N \in \mathbb{N}} \mathcal{P}(\{1, 2, \dots, N\}).$$

Question 4 Let

$$\begin{aligned} X &= \{n \in \mathbb{N} \mid \text{the last digit of } 2^n \text{ is } 6\} \\ Y &= \{n \in \mathbb{N} \mid n \text{ is divisible by } 4\}. \end{aligned}$$

Show that both sets X and Y are equal.

Question 5 (related to the Analysis course) Let S_1, S_2, S_3, S_4 be the sets of sequences $(x_n)_{n=1}^{\infty}$ of real numbers, introduced below. Decide which of these sets are subsets of the others, and which are not.

- S_1 is the set of all bounded real sequences.
- S_2 is the set of all real sequences having a limit.
- S_3 is the set of all monotone increasing real sequences.
- S_4 is the set of all monotone increasing sequences, bounded from above.

Question 6 (Ramsey's Theorem) This problem is a brainteaser. How big has a party to be that the following property is always satisfied: There are at least three people at the party who do not know each other *or* there are at least three people who know each other.

Remark: This problem could be generalised by considering the following statement: There are at least n people who do not know each other pairwise *or* there are at least n people who know each other pairwise. Ramsey's Theorem states that, for every natural $n \geq 2$, there is a finite smallest number N such that this statement holds for all parties with at least N people. Surprisingly, the exact number N is not known for $n \geq 5$.