Answers to Proof Problems

Question 1 (Proof by Induction) For $n \in \mathbb{N}$, let $A(n)$ be the following open statement:

For any n different straight lines in the plane, the regions can always be coloured black and white so that adjacent regions have different colours.

Start of Induction ($n = 1$): Any single straight line divides the plane into to halfpanes. Obviously, we can colour one halfplane black and the other white. So the statement $A(1)$ is true.

Induction Step: Assume that the statement $A(n)$ is true for some $n \in \mathbb{N}$. Consider a configuration of $n+1$ different straight lines $L_1, \ldots, L_{n+1}$ in the plane. We conclude from $A(n)$ that we can colour the regions obtained from the n lines $L_1, \ldots, L_n$ black and white so that adjacent regions have different colours. Adding the line L_{n+1} cuts through some of the regions and splits them into two, other regions are not affected. Now consider the halfplanes on both sides of the line L_{n+1}. Keep the colours of all new regions on one side of this line, and invert the colours of all new regions on the other side of this line (i.e., swap there the colours white and black). Two new adjacent regions on the same side of the line L_{n+1} will obviously have different colours, and two adjacent regions along the line L_{n+1} will also have different colours, by the construction. This shows that the statement $A(n+1)$ is also true, finishing the Induction Step.

Question 2 (Indirect Proofs)

(a) Assume that there are two numbers $x, y \in \mathbb{N}$ with $x > y$ such that $2y+1$ is not a prime but $p := x^2-y^2$ is a prime. Since we have the factorisation $x^2 - y^2 = (x-y) \cdot (x+y)$ of the prime p into two natural numbers, one of them must be 1 and the other must be p. This implies that we must have $x - y = 1$, i.e., $x = 1 + y$ and $p = x^2 - y^2 = 1 \cdot (2y+1) = 2y+1$. But $2y+1$ is not a prime number and we end up with a contradiction.

(b) Assume there exists a pair $x, y \in \mathbb{N}$ with $\sqrt{x^2+y^2} = x + y$. Squaring both sides leads to $x^2 + y^2 = (x+y)^2 = x^2 + 2xy + y^2$, i.e., $2xy = 0$. This implies that $x = 0$ or $y = 0$, in contradiction to $x, y \in \mathbb{N}$.

Question 3 (Proof by Strong Induction)

Start of Induction ($n = 12, 13, 14$): We have

$$12 = 7 \cdot 0 + 3 \cdot 4, \quad 13 = 7 \cdot 1 + 3 \cdot 2, \quad 14 = 7 \cdot 2 + 3 \cdot 0.$$

Induction Step: Assume that $n, n+1, n+2$ can be written in the form $7l + 3m$ with nonnegative integers l, m for some $n \geq 12$, in particular $n = 7l_0 + 3m_0$. Then we have

$$n + 3 = 7l_0 + 3(m_0 + 1),$$

i.e., $n + 3$ can also be written in the form $7l + 3m$ with nonnegative integers l, m.

Here is an alternative proof using normal Induction:

Start of Induction ($n = 12$): We have

$$12 = 7 \cdot 0 + 3 \cdot 4.$$

Induction Step: Assume that n can be written in the form $7l+3m$ with nonnegative integers l, m for some $n \geq 12$, in particular $n = 7l_0+3m_0$. Then we obviously have $l_0 \geq 2$ or $m_0 \geq 2$, for otherwise $7l_0 + 3m_0 \leq 7 + 3 = 10$, but $n \geq 12$. In the case $l_0 \geq 2$, we have

$$n + 1 = 7(l_0 - 2) + 3(m_0 + 5)$$

with $l_0 - 2, m_0 + 5 \in \mathbb{N} \cup \{0\}$. In the case $m_0 \geq 2$, we have

$$n + 1 = 7(l_0 + 1) + 3(m_0 - 2)$$

with $l_0 + 1, m_0 - 2 \in \mathbb{N} \cap \{0\}$, finishing the induction step in both cases.

Question 4 The flaw is in the Induction step. Let $n \in \mathbb{N}$ and k, l be natural numbers such that $\max(k, l) = n + 1$. Then it is true that $\max(k-1, l-1) = n$, but $k-1, l-1$ may no longer be both natural numbers (for example if one of k, l is equal to 1). This means that we cannot apply the Induction hypothesis to conclude that $k - 1 = l - 1$.

Question 5 (Indirect Proof) Let $x, y, z > 0$. Assume that $x > z$ and $y^2 = xz$ and not $(x > y > z)$. We need to derive a contradiction. Note first that not $(x > y > z)$ is "$x \leq y$ or $y \leq z$". We consider both cases separately:

- If $x \leq y$, then we have together with $x > z$ and the positivity of all three numbers x, y, z:
$$xz < x^2 \leq xy \leq y^2,$$
contradicting to $y^2 = xz$.

- If $y \leq z$, then we have together with $x > z$ and the positivity of all three numbers x, y, z:
$$xz > zz \geq zy \geq y^2,$$
contradicting to $y^2 = xz$.

Question 6 (Proof by Induction)

Start of Induction $(n = 0)$: $2^{3^0} + 1 = 2 + 1 = 3$ is divisible by $3^1 = 3$.

Induction Step: Assume that 3^{n+1} divides $2^{3^n} + 1$ for some $n \geq 0$. Our goal is to prove that 3^{n+2} divides $2^{3^{n+1}} + 1$. By the hint, we have
$$2^{3^{n+1}} + 1 = \left(2^{3^n}\right)^3 + 1^3 = \left(2^{3^n} + 1\right)\left(2^{2\cdot 3^n} - 2^{3^n} + 1\right).$$
We conclude from the Induction hypothesis that 3^{n+1} divides the first factor $2^{3^n} + 1$. It remains to show that 3 divides the second factor $2^{2\cdot 3^n} - 2^{3^n} + 1$. We write
$$2^{2\cdot 3^n} - 2^{3^n} + 1 = \left(2^{3^n} + 1\right)^2 - 3 \cdot 2^{3^n}.$$
Using again the Induction hypothesis, we conclude that 3 divides the right hand side of this last equation. This shows that $3^{n+1} \cdot 3$ divides $2^{3^{n+1}} + 1$, finishing the Induction Step.

Question 7 (Proof by Induction)

Start of Induction $(n = 1)$: We have $0 < a_1 = 1 < 5$.

Induction Step: Assume that $0 < a_n < 5$ for some $n \in \mathbb{N}$. Then we have, using the recursion formula and $2 < a_n + 2 < 7$,
$$a_{n+1} = \frac{(6a_n + 12) - 7}{a_n + 2} = 6 - \frac{7}{a_n + 2} < 6 - \frac{7}{7} = 5.$$
Moreover, $a_n > 0$ implies $6a_n + 5 > 0$ and $a_n + 2 > 0$, therefore
$$a_{n+1} = \frac{6a_n + 5}{a_n + 2} > 0.$$
This shows that $0 < a_{n+1} < 5$.

Question 8 (Indirect Proof) Assume that there is an integer $n \geq 2$ such that

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{1}$$

is an integer. Our goal is to show that this leads to a contradiction.

(a) Using the fact of the question, we choose a prime number p such that $p \leq n < 2p$. Next we multiply the right hand side of (1) with $a := (n!)/p$. Then every term in the new sum becomes an integer, except for the term $a\frac{1}{p}$. If $a\frac{1}{p}$ were an integer, then

$$C := 1 \cdot 2 \cdots (p-1)(p+1) \cdots (n-1)n$$

would be divisible by p. But then one of the factors of the product C would have to be divisible by p, by the unique prime factorisation. The only numbers divisible by p are the multiples of p, i.e., $p, 2p, 3p, \ldots$. But neither p is in the product C, nor any higher multiple of p, since we have $n < 2p$. This is the required contradiction.

(b) Choose the biggest integer $k \geq 0$ so that $2^k \leq n$. Then we obviously have $2^k \leq n < 2^{k+1}$. Let $D \in \mathbb{N}$ be the least common multiple of $1, 2, \ldots, n$. Using unique prime factorisation, we see that the prime factor 2 is contained in D with precisely the multiplicity k. Since $n \geq 2$, D must be even. This implies that SD is even. The fact that 2^k divides D but 2^{k+1} does not divide D implies that all terms $D \cdot \frac{1}{m}$ with $1 \leq m \leq n$, $m \neq 2^k$ are even, and that $D \cdot \frac{1}{2^k}$ is odd. But this implies that their sum SD must be odd which is a contradiction.

Remark: Since the natural numbers are involved, a first guess might be to prove the statement with Induction. But this approach will not lead to success!

Question 9 (Proof by Induction)

Start of Induction ($k = 1$): It is easy to see that if we remove a unit square anywhere in a square grid of side length 2 then we can cover the remaining area by just one piece.

Induction Step: Assume we have proved the statement for all square grids of side length 2^k. Let us consider a square grid of side length 2^{k+1}, denotes by Q. Q is obviously made up by four square grids of side length 2^k, which we denote by Q_1, Q_2, Q_3, Q_4. Assume we remove a unit square from Q, denoted by B. Assume also, without loss of generality, that this removed unit square lies in Q_1. From the Induction Hypothesis we conclude that $Q_1 \backslash B$ can be covered without overlaps by pieces. Place one piece P into the union $Q_2 \cup Q_3 \cup Q_4$, such that P covers precisely one unit square in each of the square grids Q_2, Q_3, Q_4. Using again the Induction Hypothesis, we conclude that each $Q_j \backslash P$ ($j = 2, 3, 4$) can be covered without overlaps by pieces. This shows that we can cover $Q \backslash B$ without overlaps by pieces, finishing the Induction Step. The following picture illustrates the above arguments: